Reminder:

- HW1 is due on Monday, Sept 14
- Office Hour
  
  Thursday: 11 - 12 in MATX 1118

  Friday: 2:30 - 3:30 in LSK 300

  Today only: 2:45 - 3:45 in LSK 300

- MLC (Math Learning Center) is open now.
  Hours: Mon-Fri, 11 am - 5 pm
  Location: Rooms 301 and 302 in LSK

- Take a worksheet.
Trigonometry:

Angles are an important part of trigonometric functions, so let's start with a short review about angles.

**Degree vs. Radian**

Unlike numbers, angles have a unit of measurement. There are two ways to read an angle: Degree & Radian.

**Degrees**: Cut the circle into 360 equal pieces, each piece is called 1 degree, so
- A complete circle = 360°
- Half circle = 180°
- A quarter of a circle = 90°

But the natural unit to use (and we use it in Calculus) is

**Radians**: 2π radian corresponds to an entire circle
- Half a circle : π rad
- A quarter of a circle : \( \frac{π}{2} = \frac{2π}{4} \)

**Question**: 1. What is the length of the arc of a complete circle? Its circumference = 2πr

2. What is the arc length corresponding to a slice of 1 rad.

\[
\text{1 rad} \text{ is } \frac{1}{2π} \text{ of an entire circle so the arc length is } \frac{1}{2π} \text{ of the arc of the whole circle}
\]

\[
L = \frac{1}{2π} \cdot 2πr = r
\]

*1 rad corresponds to a slice with arc length equal to 1.*
Clicker Q. \( \frac{2\pi}{3} \text{ rad} = ? \text{ degrees} \)

\[ \pi = 180^\circ, \quad 2\pi = 360^\circ \]
\[ \Rightarrow \quad \frac{2\pi}{3} = \frac{360^\circ}{3} = 120^\circ \]

A. 135°
B. 45°
C. 60°
D. 120°

\[ \Rightarrow [D] \]

Convert degrees \( \rightarrow \) radian

\[ 135^\circ \rightarrow 135 \times \frac{\pi}{180} = \frac{3\pi}{4} \]

\[ \theta = \text{angle in deg} \rightarrow \theta^\circ \times \frac{\pi}{180} = \theta \text{ in rad} \]

\[ \theta^\circ = \theta \text{ in rad} \times \frac{180}{\pi} \]

\[ \frac{11\pi}{6} \text{ rad} \rightarrow \frac{11\pi}{6} \times \frac{180}{\pi} = 330^\circ \]

* Important to remember: know the following common angles in radians.

<table>
<thead>
<tr>
<th>Degree</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radian</td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\pi}{2} )</td>
<td>( \pi )</td>
<td>( \frac{3\pi}{2} )</td>
<td>( 2\pi )</td>
</tr>
</tbody>
</table>

* Note that when there is no unit is given for the angle, it is assumed that the unit is radian.

**Trig Ratios:** They’re all about a right triangle and the relation b/w the lengths of its sides and the angles.

\[ \cos \alpha = \frac{\text{adj}}{\text{hyp}} \]
\[ \sin \alpha = \frac{\text{opp}}{\text{hyp}} \]
\[ \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\text{opp}}{\text{adj}} \]
\[ \cot \alpha = \frac{1}{\tan \alpha} = \frac{\cos \alpha}{\sin \alpha} \]
\[ \sec \alpha = \frac{1}{\cos \alpha} \]
\[ \csc \alpha = \frac{1}{\sin \alpha} \]

* adj and opp may change depending on where we choose angle \( \alpha \).
Remark: Note that if we change the size of the right triangle but keep the angle \( \alpha \) unchanged, the trig ratios remain fixed: \( \frac{b}{c} = \frac{B}{C} = \frac{\beta}{\gamma} = \sin \alpha \) and so on for all others.

In all of these triangles, \( \sin \alpha, \cos \alpha, \tan \alpha, \ldots \) are all the same even though the side lengths are changed.

Trigs most useful tool:

The Unit Circle

Consider a right triangle whose hypotenuse = 1, and also consider a circle centred at the origin with radius = 1.

Place the triangle inside the circle such that its hypotenuse becomes the radius of the circle.

From the unit circle, we get:

\[ -1 \leq \cos \theta \leq 1, \quad -1 \leq \sin \theta \leq 1 \]

Rotating on the circle for more than one cycle, does NOT change \( \sin \) or \( \cos \); just angles get bigger or smaller.
Read $\sin \theta$ and $\cos \theta$ in the unit circle:

* The positive $x$-axis is where the angle starts.
* Go counter-clockwise for $+$ angle and clock-wise for $-$ angle.
* Find the radius of the circle corresponding to angle $\theta$ and the point $P$ where the radius intersects the circle.
* The $x$-coordinate of $P = \cos \theta$ \quad \Rightarrow \quad P = (x, y)$
* The $y$-coordinate of $P = \sin \theta$ \quad \Rightarrow \quad P = (\cos \theta, \sin \theta)$

For example, let's find angles $\theta = \frac{2\pi}{3}, \frac{7\pi}{6}, -\frac{\pi}{4}$

For $\frac{2\pi}{3}$, which quadrant?
\[ \frac{2\pi}{3} = \frac{360}{3} = 120^\circ \]

Better way: Break the angle
\[ \frac{2\pi}{3} = \frac{3\pi - \pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \pi - \frac{\pi}{3} \] 2nd quadrant

\[ \theta = \frac{7\pi}{6} = \frac{6\pi + \pi}{6} = \pi + \frac{\pi}{6} \]

3rd quad

\[ \theta = -\frac{\pi}{4} \]

Reference

Sin/Cos of Common Angles:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0\rightarrow 30^\circ</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

\[ \frac{\sin x}{\cos x} = \tan x \]  

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{\sqrt{3}} \rightarrow 30^\circ$</td>
<td>$\sqrt{3} \rightarrow 60^\circ$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\frac{\sqrt{3}}{3}$</td>
</tr>
</tbody>
</table>

\[ \frac{\sin x}{\cos x} = \tan x \]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0 \rightarrow 30^\circ$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>$0$</td>
</tr>
<tr>
<td>$x$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>-------</td>
<td>----------</td>
</tr>
<tr>
<td>$0$</td>
<td>$0 \rightarrow 45^\circ$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

$=\text{undefined}$
Graph of Sin and Cos:

Take the angle as the input, then we can construct trig functions.

\[
\theta \rightarrow \sin \theta \quad \text{numbers} \quad \alpha \rightarrow \tan \alpha \quad \text{numbers}
\]

* Whatever the notation is, we keep in mind that the input \((\theta, x, t, \ldots)\) is an angle in radian and output is ....

We can use the table (last page) to sketch the graph of

\[
f(x) = \sin x \quad \text{and} \quad f(x) = \cos x
\]

Observations

* Domain: \(\mathbb{R}\) (for both)
* Range: \([-1,1]\) \(\Rightarrow\) Sin and Cos NEVER return a number greater than 1 or less than -1.
* \(\sin x\) and \(\cos x\) repeat themselves every \(2\pi\).
  
  4 periodic functions with period = \(2\pi\) i.e.
  
  \[
  \sin(\theta) = \sin(\theta + 2\pi)
  \]
  
  \[
  \cos(\theta) = \cos(\theta + 2\pi)
  \]
Clicker Q: Evaluate $\sin\left(\frac{4\pi}{3}\right)$ and $\cos\left(\frac{4\pi}{3}\right)$ (respectively)

A. $\frac{\sqrt{3}}{2}, \frac{1}{2}$  
B. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}$  
C. $-\frac{1}{2}, \frac{\sqrt{3}}{2}$  
D. $-\frac{1}{2}, -\frac{\sqrt{3}}{2}$

1st step: Locate $\frac{4\pi}{3}$ in the unit circle.

$\theta = \frac{4\pi}{3}$ = $\frac{3\pi + \pi}{3}$ = $\pi + \frac{\pi}{3}$

Table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Quadrant</th>
<th>Sin</th>
<th>Cos</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/3$</td>
<td>3rd</td>
<td>$\sqrt{3}/2$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$4\pi/3$</td>
<td>3rd</td>
<td>$-\sqrt{3}/2$</td>
<td>$-1/2$</td>
</tr>
</tbody>
</table>

$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \rightarrow [B]$

$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

Question 1: Find an angle $\theta$ for which $\sin \theta = \sin \frac{4\pi}{3}$

Question 2: Find an angle $\alpha$ for which $\cos \alpha = \cos \frac{4\pi}{3}$

* The angle in red has the same Sin value as $4\pi/3$

$\theta = 2\pi - \frac{\pi}{3} = \frac{6\pi - \pi}{3} = \frac{5\pi}{3}$

* The blue angle has the same Cos value as $4\pi/3$

$\alpha = \pi - \frac{\pi}{3} = \frac{3\pi - \pi}{3} = \frac{2\pi}{3}$

Practice Questions: Find $\sin$, $\cos$, $\tan$ and $\sec$ of the given angle.

(a) $-\frac{\pi}{3}$  
(b) $\frac{7\pi}{3}$  
(c) $\frac{17\pi}{2}$  
(d) $\frac{19\pi}{6}$  
(e) $\frac{5\pi}{6}$  
(f) $\frac{-2\pi}{3}$  
(g) $-\frac{5\pi}{4}$