Last class: \( y = f(x) \) a given function

→ Derivative of \( f(x) \): the slope of the tangent line to the graph of \( f(x) \) at any \( x \)

→ Math definition of the derivative:

\[
\frac{df}{dx} \quad \text{or} \quad y'
\]

→ Notation:

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

Examples:

(1) \( f(x) = x^2 \quad \Rightarrow \quad f'(x) = ? = 2x \)

\[
\begin{align*}
\frac{df}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{h(2x+h)}{h} = 2x
\end{align*}
\]

(2) \( f(x) = x^3 \quad \Rightarrow \quad f'(x) = ? = 3x^2 \)

\[
\begin{align*}
\frac{df}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= 3x^2
\end{align*}
\]

For each \( x \) value, find \( m_{\text{tan}} \) at that \( x \) value and make a graph for \( m_{\text{tan}} \) or for \( f(x) \).
Graph of \( f(x) = x^3 \) and its derivative \( f'(x) = 3x^2 \).

* At \( x = 0 \) \( m_{\text{tan}} = 0 = f'(0) \) \( \Rightarrow \) graph of \( f \) crosses \( x \)-axis at 0.
* Everywhere else \( m_{\text{tan}} > 0 \) \( \Rightarrow \) \( f'(x) > 0 \) \( \Rightarrow \) graph of \( f \) above \( x \)-axis.

**Example 1:** Using the fact that \( f'(x) \) is the slope of the tangent line, sketch the graph of \( f'(x) \) for the given function \( f(x) \).

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**Clicker Q:** Which one is the graph of \( f'(x) \) for the following \( f \).
Relationship between \( f \) and \( f' \) graphically:

If \( f \) is increasing \( \iff \) \( m_{\text{tan}} > 0 \iff \) \( f' \) positive
above \( x \)-axis

If \( f \) is decreasing \( \iff \) \( m_{\text{tan}} < 0 \iff \) \( f' \) negative
below \( x \)-axis

If \( f \) is constant \( \iff \) \( f \) has a horizontal \( \iff m_{\text{tan}} = f' = 0 \)
tangent line

Example 2. Find the equation of the tangent line to \( f(x) = x^2 \) at
\( x = 3 \).

*This line is the tangent line so
\( m = m_{\text{tan}} \) at \( 3 = f'(3) \)

point \( \rightarrow \) tangency \( \rightarrow \) point: \( (3, f(3)) = (3, 9) \)

We already know if \( f(x) = x^2 \) then \( f'(x) = 2x \) so
\( m_{\text{tan}} \) at \( 3 = f'(3) = 2 \cdot 3 = 6 \)

\( m_{\text{tan}} = 6 \)
point: \( (3, 9) \)
\( y - y_0 = m(x - x_0) \Rightarrow y - 9 = 6(x - 3) \)
\( \Rightarrow y = 6x - 9 \)
Example 3: Find the equation of the tangent line to the graph of the function \( f(x) = \frac{x}{x-3} \) at \( x = 4 \).

We need \( m_{\text{tan}} \) at 4 and a point:

\[ m_{\text{tan}} = f'(x) \quad \xrightarrow{x=4} \quad m_{\text{tan}} = f'(4) \]

Tangency point: \( (4, f(4)) \)

\[ f(4) = \frac{4}{4-3} = \frac{4}{1} = 4 \quad \Rightarrow \quad \text{point:} \quad (4, 4) \]

How to find \( f'(4) \)? For now, use the limit definition of the derivative.

\[
f(x) = \frac{x}{x-3} \quad \Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
\xrightarrow{x=4} \quad f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}
\]

Plug in \( 4+h \) and 4 into \( f(x) \):

\[
= \lim_{h \to 0} \frac{(4+h-3) - 4}{h}
\]

Simplify:

\[
= \lim_{h \to 0} \frac{4+h - 4(1+h)}{1+h}
\]

Take the common denominator:

\[
= \lim_{h \to 0} \frac{4+h - 4 - 4h}{1+h}
\]

Simplify:

\[
= \lim_{h \to 0} \frac{-3h}{1+h} \cdot \frac{1}{h} = -3
\]

Cancel \( x \) re-sub:

\[
= \lim_{h \to 0} \frac{-3h}{1+h} \cdot \frac{1}{h} = -3
\]
So we found 

\[ f'(4) = -3 = m \tan \text{tangency point } (4,4) \]

\[ y - y_0 = m(x - x_0) \]

\[ \Rightarrow y - 4 = -3(x - 4) \Rightarrow y = -3x + 16 \]