Review of Functions

- **Polynomials** \(\rightarrow\) **Linear Functions**: \(y = mx + b\) or \(y - y_0 = m(x - x_0)\)
  
  Domain = \(\mathbb{R}\)

- **Quadratic Functions**: \(y = ax^2 + bx + c\)
  
  \(a > 0\) \(\bigguparrow\) \(a < 0\) \(\biggdownarrow\)
  
  Vertex: \((-\frac{b}{2a}, f(-\frac{b}{2a}))\)
  
  Higher degrees:
  
  \(y = x^3 + 5x, y = 2x^4 + 5x^6 \ldots\)

Solving quadratic equations by quadratic formula:

\(y = ax^2 + bx + c \implies x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\)

- First look for factoring, then try quadratic formula.
- Practice factoring, we need it in limits as well.

- **Rational Functions**: \(\frac{\text{Polynomial}}{\text{polynomial}} = \frac{5x^2 + 3x + 1}{10x^4 + x^2}\)
  
  Domain = \(\mathbb{R}\) - \(\{\text{roots of the denominator}\}\)

  Important example: \(y = \frac{1}{x}\)
  
  *Remember the graph & its asymptotic behaviour.

- **Root Functions (square roots)**: \(\sqrt{\phantom{0}}\) : Domain: \(\bigcirc \bigguparrow 0\)
  
  *If square root sits in the denominator we don't consider \(\bigcirc = 0\).

  Important Example: \(y = \sqrt{x}\)

- **Trig Functions**
  
  * Domain = \(\mathbb{R}\) \(\bigguparrow\) \(y = \sin x\)
  
  * Everything in Radian \(\bigguparrow\) \(y = \cos x\)
  
  * Range = \([-1, 1]\)

  * Remember \(\sin \) & \(\cos \) of common angles: \((0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, 2\pi, 2\pi)\)

  * Remember the unit circle

  \[
  \frac{7\pi}{6} = \frac{6\pi + \pi}{6} = \pi + \frac{\pi}{6}
  \]

  \[
  \sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}
  \]
- **Exponential function** \( y = b^x \) \((b > 0, b \neq 1)\)

  - Very important: \( y = e^x \)
    - Domain: \( \mathbb{R} \)
    - Range: \((0, \infty)\)
    - \(e^x \neq 0\) and \(e^x\) is NEVER negative.

- **Logarithm function**
  \[ y = \log_b x \quad \text{if} \quad b = e \quad \Rightarrow \quad y = \log_e x = \ln x \]

  - Domain: \((0, \infty)\)
  - Range: \(\mathbb{R}\)

*Remember these important graphs, they will help you a lot especially when finding V.A. and H.A. of these functions. Also, you should know how to solve exponential & logarithmic equations.

### Limits

- Graphically: approach a value on the x-axis, track the graph and find the y-value to which the graph approaches.

- Computationally:
  - **Step I:** Start with substitution
  - **Step II:** You get
    - If you get \(0/0\), then factor
      - Conjugate multiplication
      - Sub again
    - Else, simplify
      - Common denominator
      - Common denominator
  - **Step III:** You get
    - Nonzero number
    - \(1/0\)
    - \(\pm\infty\)
    - \(0^+/0^-\)
    - Finite number \(L\)
    - \(\infty/\infty\)
  - **Step IV:** Find the dominant term, cancel any common terms, and argue if you have
    - \(\infty/\infty\)
    - \(\infty/\text{finite}\)
    - \(\text{finite}/\text{finite}\)

*In the diagram above*

- **Case III:** Vertical asymptote
  - \(\lim_{x \to a} f(x) = \pm \infty\)
  - \(x = a\) is V.A.

- **Case IV:** Horizontal asymptote
  - \(\lim_{x \to \pm \infty} f(x) = b\)
  - \(y = b\) is HA

*You MUST verify each limit and show limit computation.
Derivative → Slope of the tangent line:

\[ f(x) = \frac{1}{2x} \rightarrow f(x+h) = \frac{1}{2x} + h \]

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{2(x+h)} \]

- \( x+h \) sits for all \( x \) given in the function \( f \)
- Use brackets when substituting \( x+h \)
- Use brackets for the 2nd term so that you remember to distribute signs in all the terms following it.

* Read the question carefully:
  → if \( f'(x) \) then use the formula but
  → if \( f'(3) \) for example then \( x=3 \) so \( f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \)

\[ \text{easier computation.} \]

Derivative Rules:

- **Power rule:** \( (x^n)' = nx^{n-1} \)
- **Constant function:** \( (c)' = 0 \)
- **Trigs:**
  - \( (\cos x)' = -\sin x \)
  - \( (\sin x)' = \cos x \)
  - \( (\tan x)' = \sec^2 x \)
- **Exp:**
  - \( (b^x)' = b^x \ln b \)
  - \( (e^x)' = e^x \)
- **Ln:** \( (\ln x)' = \frac{1}{x} \)

Combination of functions:

- \( (f \pm g)' = f' \pm g' \)
- **Constant multiple:** \( (cf(x))' = cf'(x) \)
- **Product:** \( (fg)' = fg' + f'g \)
- **Quotient:** \( \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \)
- Chain rule & functions of functions:
  - \( (f(g(x)))' = f'(g(x)) \cdot g'(x) \)
  - \( (\sin(\cos x))' = (\cos x) \cdot (-\sin x) \)
  - \( (e^{\cos x})' = (-\sin x) \cdot e^{\cos x} \)
  - \( (e^{x^2})' = 2xe^{x^2} \)

* All these rules can be combined together.
* Can be used multiple times in one function:
  - \( \left( \frac{x+1}{x^2+3} \right)' \)
  - \( \left( \frac{\sin x}{x+1} \right)' \)
  - \( \ln(\sin x) \)
  - \( \sqrt{x^2+x} \)
Equation of the tangent line

\[ y - y_0 = m(x - x_0) \]

- comes from \( m_{\text{tan}} = f'(x) \) evaluated at the tangency point \((x_0, y_0)\) are the coordinates of the tangency point.
- Plug \( x_0 \) into the original function \( f(x) \) and find \( y_0 \)

Relation between \( f \) and \( f' \) graphically

* Always remember \( f' \): slope of the tangent line

\[
\left( \frac{1}{2x} \right)' = -\frac{1}{(2x)^2} \quad \text{so} \quad f' \circ \rightarrow m_{\text{tan}} \circ \quad \text{tangent line} \quad \text{curve of } f
\]

Always - \( f' \circ \rightarrow m_{\text{tan}} \circ \quad \text{curve of } f \)

Asymptotes of important function:

\[ y = e^x \]

- \( \lim_{x \to -\infty} e^x = 0 \Rightarrow y = 0 \) H.A. (left)
- \( \lim_{x \to \infty} e^x = \infty \)

Similar for \( e^{-x} \)

- \( e^x \) has NO V.A.: defined everywhere

\[ y = \ln x \]

- \( \ln 0 \) is undefined \( \Rightarrow \lim_{x \to 0^+} \ln x = -\infty \Rightarrow x = 0 \) V.A. (downward)
- \( \lim_{x \to \infty} \ln x = \infty \) NO H.A.
Practice Problems

1. Evaluate the following: 
   \(\sin\left(\frac{3\pi}{6}\right), \cos\left(-\frac{\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right), \sin\left(\frac{11\pi}{3}\right), \cos\left(\frac{7\pi}{6}\right)\)

2. What limit strategy do you use for each of the following? Find each limit:

   (a) \(\lim_{x \to 2} \frac{x^2 - 6x + 8}{x - 1}\) 
      \[\text{Substitution} \quad \frac{a - e}{c} \rightarrow C\]

   (b) \(\lim_{x \to 2} \frac{x^2 - 6x + 8}{x - 2}\) 
      \[O \quad \text{factor} \quad \frac{f - i}{g} \rightarrow g\]

   (c) \(\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}\) 
      \[\text{conjugate} \quad \frac{\sqrt{2-x} + 1}{\sqrt{2-x} + 1}\]

   (d) \(\lim_{x \to 1} \frac{\sqrt{2-x} - 1}{x - 1}\)
      \[\text{split into} \quad x \to 4^+, \quad x \to 4^- \rightarrow \text{DNE}\]

   (e) \(\lim_{x \to 4} \frac{|x - 4|}{x - 4}\)
      \[\text{split into} \quad x \to 4^+, \quad x \to 4^- \rightarrow \text{DNE}\]

   (f) \(\lim_{x \to 1} \frac{x^2 - 2x + 1}{x - 1}\)
      \[\begin{cases} 
          x^2 - 2x + 1 & x \leq 1 \\
          \ln x & x > 1 
        \end{cases}\]

   (g) \(\lim_{x \to \frac{\pi}{2}^+} \frac{\sin x}{x - \frac{\pi}{2}}\)

   (h) \(\lim_{x \to \infty} \frac{x^2 - 3x + 1}{x^2 + 2}\) 
      \[\approx \frac{x^2}{x^2} = \frac{1}{1} = 1 \rightarrow 0 \quad (x \to \infty)\]

   (i) \(\lim_{x \to \infty} \frac{x^2}{x^2 + 3x + 1}\) 
      \[\approx \frac{x^2}{x^2} = \frac{1}{1} = 1 \rightarrow \infty \Rightarrow y = 0 \text{ H.A.}\]

3. Find V.A. and H.A. of the function \(f(x) = \frac{x^3 - 3x^2 + 1}{2x^3 - 8x}\)

4. Find H.A. of \(f(x) = \frac{1}{e^x + e^{-x}}\)

5. Find V.A. of \(f(x) = \frac{\sin x}{x + 2}\)
   
   \[\sin x \quad \text{is a candidate} \quad x = -2\]
   \[\begin{align*}
   \lim_{x \to -2^+} \frac{\sin x}{x + 2} &= \frac{\sin(2)}{2^+} = 0 \\
   \lim_{x \to -2^-} \frac{\sin x}{x + 2} &= \frac{\sin(-2)}{2^-} = \text{nonzero} \\
   \lim_{x \to -2} \frac{\sin x}{x + 2} &= -\infty
   \end{align*}\]
6. Differentiate the following functions.
   (a) \( e^x \cos x \)
   (b) \( \frac{\sin x}{3(x+1)} \)
   (c) \( \sqrt{e^{2x} + \cos(x+x^{1/3})} \)
   (d) \( \cos(\ln x) \)
   (e) \( 2^x \cdot e^{5x^2+x} \)

7(i). Find the derivative of the following functions using the limit definition.
   (a) \( f(x) = \frac{2x}{x-1} \)
   (b) \( \frac{2}{x} \)
   (c) \( \sqrt{2x-3} \)
   (d) \( x^2 + x \)

(ii). Find the derivatives of (a)-(d) using another method of your choice.

(iii). Find the equation of the tangent line to the graph of function (b) at \( x = -1 \) and sketch the graph & its tangent line.

(iv). T/F: function (a) is always increasing.
    Function (b) is always decreasing.
    Function (d) is always increasing.

8. Find all \( x \) values where the following function has a horizontal tangent line.
   \[ f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1800 \]

9. Sketch a function satisfying the following conditions.
   - Domain of \( f \) is \( \mathbb{R} \) except \( -4 \leq x \leq 4 \)
   - \( f(x) \) has a V.A. at \( x = -1 \)
   - \( \lim_{x \to -1^+} f(x) = \infty \)
   - \( \lim_{x \to 1^-} f(x) \) does not exist
   - \( f''(3) = 0 \)
   - \( f'(-3) < 0 \)
   - \( \tan < 0 \)
(c) \[ \lim_{x \to 3} \frac{1 \cdot 3 - 1 \cdot x}{3x - 3} = \frac{1}{3} - \frac{1}{3} = 0 \]

\[ \lim_{x \to 3} \frac{3 - x}{3x - 3} = \lim_{x \to 3} \frac{3 - x}{3x} \frac{1}{1 - x/3} = \frac{1}{3x} = \frac{-1}{9} \]

\[ 3 - x = -(-3) \]
\[ x = -(-3) \]
\[ = -1(-3 + x) \]
\[ \frac{1}{x - 3} \cdot \frac{1}{(x - 3)} = \lim_{x \to 3} \frac{-1}{3x} = \frac{-1}{9} \]

\[ \lim_{x \to \frac{\pi}{2}^+} \frac{\sin x}{x - \frac{\pi}{2}} = \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2} - \frac{\pi}{2}} = \frac{1}{0^+} = +\infty \]

\[ \Rightarrow \quad x = \frac{\pi}{2} \quad V. A. \]
(4) \( f(x) = \frac{1}{e^x + e^{-x}} \)

\[
\lim_{{x \to \infty}} f(x) = \lim_{{x \to \infty}} \frac{1}{e^x + e^{-x}} = \frac{1}{\infty + 0} = \frac{1}{\infty} = 0
\]

\[x \to \infty : \quad e^x \to \infty\]
\[x \to \infty : \quad e^{-x} \to 0\]

\[
\lim_{{x \to -\infty}} f(x) = \lim_{{x \to \infty}} \frac{1}{e^x + e^{-x}} = \frac{1}{0 + \infty} = \frac{1}{\infty} = 0
\]

\[x \to -\infty : \quad e^x \to 0\]
\[x \to -\infty : \quad e^{-x} \to \infty \quad \Rightarrow \quad y = 0 \quad H.A.\]