Last class: Derivative of \( f(x) \)

Slope of the tangent line to the graph of \( f(x) \) at any \( x \)

\[
\text{limit definition:} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\( \rightarrow \) Equation of the tangent line to the graph of \( y = f(x) \) at \( x = a \)

Recall:

\[
\text{To write the equation of a line we need two things:} \quad \text{Slope} \quad \text{a point on the line} \quad (x_0, y_0)
\]

\[
y - y_0 = m(x - x_0)
\]

Since this line is tangent we can find its slope \( m_{\text{tan}} \) by evaluating the derivative at \( a \)

\[
m_{\text{tan}} = f'(a)
\]

and the point on the line is the tangency point where \( f(x) \) and the line touch: \( (a, f(a)) \)

so the equation of the tangent line is:

\[
y - f(a) = f'(a)(x-a) \quad \rightarrow \text{important}
\]

* Note that the fact that \( (a, f(a)) \) is both on the line and on the function helps us to get info about line and function both.
Derivatives we computed so far:
\[ f(x) = x^2 \implies f'(x) = 2x \]
\[ f(x) = x^3 \implies f'(x) = 3x^2 \]

Now you guess:
\[ f(x) = x^4 \implies f'(x) = 4x^3 \]
\[ f(x) = x^9 \implies f'(x) = 9x^8 \]
\[ \vdots \]
\[ f(x) = x^{100} \implies f'(x) = 100x^{99} \]

In general, for any real number \( n \):
\[ f(x) = x^n \implies f'(x) = nx^{n-1} \]

The rigorous mathematical way to show that the power rule is really true is by using the limit definition of the derivative. One can verify that:

\[ f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

for example:
\[ f(x) = \frac{1}{x} = x^{-1} \]

Recall:
\[ \frac{1}{x^n} = x^{-n} \]

\[ f'(x) = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2} \]

Recall:
\[ \sqrt[n]{x} = x^{\frac{m}{n}} \]

Example 1: Use the power rule to find \( f'(x) \).

\begin{enumerate}
\item a) \( f(x) = \sqrt{x} = x^{\frac{1}{2}} \)
   \[ f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}} \]
   \[ f'(x) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \]
\item b) \( f(x) = \frac{1}{x} = x^{-1} \)
   Recall:
   \[ \frac{1}{x^n} = x^{-n} \]
   \[ f'(x) = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2} \]
\item c) \( f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}} \)
   \[ f'(x) = \frac{2}{3} \cdot x^{\frac{2}{3} - 1} = \frac{2}{3} \cdot x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}} \]
\item e) \( f(x) = x^\pi \)
   \[ f'(x) = \pi x^{\pi - 1} \]
\end{enumerate}
What is the rule for the trig functions $\sin x$ and $\cos x$?

Recall the graph of $y = \sin x$:

- $f(-\frac{3\pi}{2}) = 0 \iff m_{\tan} = 0 \text{ at } x = -\frac{3\pi}{2}$
- $f(-\pi) = -1$
- $m_{\tan} = 1 \text{ at } x = \frac{\pi}{2} \Rightarrow f'(\frac{\pi}{2}) = 0$
- $m_{\tan} = 0 \text{ at } x = \frac{\pi}{2} \Rightarrow f'(\frac{\pi}{2}) = 0$
- $m_{\tan} = -1 \text{ at } x = \pi \Rightarrow f'(\pi) = -1$
- $m_{\tan} = 1 \text{ at } x = \pi \Rightarrow f'(\pi) = 1$
- $m_{\tan} = 0 \text{ at } x = \frac{3\pi}{2} \Rightarrow f'(\frac{3\pi}{2}) = 0$
- $m_{\tan} = 0 \text{ at } x = \frac{3\pi}{2} \Rightarrow f'(\frac{3\pi}{2}) = 0$
- $m_{\tan} = 1 \text{ at } x = 0 \Rightarrow f'(0) = 1$

We track $m_{\tan}$ at different points and we'll get a graph for $f'(x)$:

Look familiar?

Yes, this is the graph of $\cos x$:

$f(x) = \sin x \implies f'(x) = \cos x$

In fact, one can rigorously verify this by applying the limit definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

bunch of algebra with trig identities, etc.

$$= \cos x$$
We can do a similar process for

\[
\frac{d}{dx} \cos x \implies \frac{d}{dx} \cos x = -\sin x
\]

and

\[
\frac{d}{dx} \tan x \implies \frac{d}{dx} \tan x = 1 + \tan^2 x = \sec^2 x = \frac{1}{\cos^2 x}
\]

Now let's go to the derivative of another family of functions:

**Exponential functions:**

\[
\frac{d}{dx} 2^x \implies \frac{d}{dx} 2^x = 2^x \ln 2
\]

2^x is an exponential function not a power function so power rule does not apply.

\[
\frac{d}{dx} 3^x \implies \frac{d}{dx} 3^x = 3^x \ln 3
\]

Similarly:

\[
\frac{d}{dx} e^x \implies \frac{d}{dx} e^x = e^x \ln e = e^x
\]

In general:

\[
\frac{d}{dx} b^x \implies \frac{d}{dx} b^x = b^x \ln b
\]

\(b > 0\)

Compare:

\[
\frac{d}{dx} x^2 = 2x \quad \frac{d}{dx} x^n = nx^{n-1}
\]

\[
\frac{d}{dx} 2^x = 2^x \ln 2 \quad \frac{d}{dx} \pi^x = \pi^x \ln \pi
\]
Clicker Q: What is the equation of the tangent line to 
\( f(x) = e^x \) at \( x = 0 \)?

A. \( y = x \)
B. \( y = x + 1 \)
C. \( y = x - 1 \)
D. \( y = e^x + 1 \)

\[ f(x) = e^x \quad \text{at} \quad x = 0 \]

We need slope at \( x = 0 \) and the tangency point.

\[ m_{\text{tan}} \quad \text{at} \quad x = 0 : f'(0) \]
\[ \text{tangency point} : \quad x = 0 \]
\[ y = e^0 = 1 \]

\[ f'(x) = e^x \quad \text{at} \quad x = 0 \quad \Rightarrow f'(0) = e^0 = 1 = m_{\text{tan}} \]
\[ \Rightarrow y - y_0 = m(x - x_0) \Rightarrow y - 1 = 1(x - 0) \Rightarrow y = x + 1 \]

Practice: Find the equation of the tangent line to \( f(x) = \cos x \)

at \( x = \frac{\pi}{2} \). Sketch \( f(x) \) and its tangent line at \( x = \frac{\pi}{2} \).

Clicker Q: \( f(x) = x^3 + \sin x \), \( f'(x) = ? \)

A. \( 3x^2 + \cos x \)
B. \( 3x^2 + \sin x \)
C. \( x^3 + \cos x \)
D. \( 3x^2 - \sin x \)

\((x^3)' = 3x^2\) and \((\sin x)' = \cos x \Rightarrow (x^3 + \sin x)' = 3x^2 + \cos x\)

In general:
\[
(f(x) + g(x))' = f'(x) + g'(x)
\]
\[
(f(x) - g(x))' = f'(x) - g'(x)
\]
Clicker Q: \( f(x) = 5 \), \( f'(x) = ? \)

A. 5  
B. 0  
C. undefined  
D. Don’t know

\[ f(x) = 5 \quad \Rightarrow \quad m_{\text{tan}} = 0 \quad \text{for all } x \]
\[ \Rightarrow \quad f'(x) = 0 \]

**In general**

\[ f(x) = C \quad \Rightarrow \quad f'(x) = 0 \]

Clicker Q: \( f(x) = 5 \sin x \), \( f'(x) = ? = 5 \cos x \)

A. 5  
B. 0  
C. 5 \cos x  
D. \cos x

**In general**

\[ f(x) = C \cdot g(x) \quad \Rightarrow \quad f'(x) = C \cdot g'(x) \]

Practice: Find \( f'(x) \).

(a) \( f(x) = 4x^2 - 3x + 2 \)
(b) \( f(x) = 5 \cos x + 2^x + 5 \)
(c) \( f(x) = 3e^x + \sqrt{x} + \frac{3}{x^2} - 10 \)
(d) \( f(x) = x^{11} - \pi^x - \frac{x}{3} + \frac{3}{x} \)
(a) \[ f(x) = 4x^2 - 3x + 2 \]

\[ (4x^2)' = 4(x^2)' = 4 \cdot 2x = 8x \]

\[ (3x)' = 3(x)' = 3 \cdot 1 = 3 \]

\[ \Rightarrow \quad f'(x) = 8x - 3 \]

\[ (2)' = 0 \]

(b) \[ f(x) = 5 \cos x + 2^x + 5 \]

\[ (5 \cos x)' = 5(\cos x)' = 5(-\sin x) \]

\[ (2^x)' = 2^x \cdot \ln 2 \]

\[ (5)' = 0 \]

\[ \Rightarrow \quad f'(x) = -5 \sin x + 2^x \cdot \ln 2 \]

(c) \[ f(x) = 3e^x + \frac{3}{\sqrt{x}} + \frac{3}{x^2} - 10 \]

\[ (3e^x)' = 3(e^x)' = 3e^x \]

\[ (\frac{3}{\sqrt{x}})' = (\frac{1}{3})' = \frac{1}{3} \cdot \frac{-2}{3} = \frac{1}{3\sqrt[3]{x^2}} \]

\[ (\frac{3}{x^2})' = (3x^{-2})' = 3(x^{-2})' = 3 \cdot (-2)x^{-3} = -\frac{6}{x^3} \]

\[ (10)' = 0 \]

\[ \Rightarrow \quad f'(x) = 3e^x + \frac{1}{3\sqrt[3]{x^2}} - \frac{6}{x^3} \]

(d) \[ f(x) = x^{\pi} - \pi^x - \frac{x}{3} + \frac{3}{x} \]

\[ (x^{\pi})' = \pi x^{\pi - 1} \]

\[ (\pi^x) = \pi^x \cdot \ln \pi \]

\[ (\frac{x}{3})' = (\frac{1}{3}x)' = \frac{1}{3} (x)' = \frac{1}{3} \]

\[ (\frac{3}{x})' = (3x^{-1})' = 3(x^{-1})' = 3 \cdot (-1)x^{-2} = -\frac{3}{x^2} \]

\[ \Rightarrow \quad f'(x) = \pi x^{\pi - 1} - \pi^x \cdot \ln \pi - \frac{1}{3} - \frac{3}{x^2} \]