To find \( \int_{a}^{b} f(x) \, dx \), we find the anti-derivative of \( f(x) \) first; i.e., \( F(x) \) such that \( F'(x) = f(x) \) and we evaluate the values \( F(b) \) and \( F(a) \) then we use

\[
FTC : \quad \int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

**Examples:**
This is called the definite integral of \( f(x) \), because the lower and upper bounds of integral, i.e., \( a \) & \( b \) are given

1. \[
\int_{0}^{2} x^2 \, dx = F(2) - F(0) = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}
\]

Find a function \( F(x) \) such that:

\[
F'(x) = x^2 \quad \Rightarrow \quad F(x) = \frac{x^3}{3} = \frac{1}{3} x^3
\]

Shortcut notation for \( F(3) - F(-1) \)

\[
\left. \frac{1}{4} x^4 \right|_{x=-1}^{x=3} = \frac{1}{4} (3)^4 - \frac{1}{4} (-1)^4 = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20
\]

2. \[
\int_{-1}^{3} x^3 \, dx = \frac{1}{4} x^4 \bigg|_{x=-1}^{x=3} = \frac{1}{4} (3)^4 - \frac{1}{4} (-1)^4 = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20
\]

* You may start by

\[
F(x) = \frac{x^3}{3}
\]

then derive:

\[
F(x) = 3x^2 \quad \Rightarrow \quad NOT \; quite \; right
\]

divide by 3

\[
F(x) = \frac{x^3}{3} \quad \Rightarrow \quad F'(x) = \frac{1}{3} \cdot 3x^2 = x^2
\]

**In general:**

Power rule for integration

\[
\text{If } f(x) = x^n \text{ then } \quad F(x) = \frac{1}{n+1} x^{n+1}
\]

\[n \neq -1\]

\[\Rightarrow \text{Think: Is this the "only" } \quad F(x) ?\]
\[
\int_1^2 \left( \sqrt{x} + 3x^5 - \frac{1}{\sqrt[3]{x^2}} \right) \, dx
\]

First rewrite the integrand in power form, and break the integral into 3 integrals:

\[
\int_1^2 x^{\frac{1}{2}} \, dx + \int_1^2 3x^5 \, dx - \int_1^2 x^{-\frac{2}{3}} \, dx
\]

\[
= \left( \frac{1}{\frac{1}{2} + 1} \right) \bigg|_{x=1}^{x=2} + 3 \left( \frac{1}{5 + 1} \right) \bigg|_{x=1}^{x=2} - \left( \frac{1}{-\frac{2}{3} + 1} \right) \bigg|_{x=1}^{x=2}
\]

be careful!

Simplify then

Use FTC & substitute the bounds

\[
\int_1^2 x^{\frac{1}{2}} \, dx = \frac{1}{\frac{3}{2}} \bigg|_{x=1}^{x=2} = \frac{3}{2} \bigg|_{x=1}^{x=2} = \frac{3}{2} - \frac{1}{2} = 1
\]

\[
\int_1^2 3x^5 \, dx = 3 \left( \frac{1}{6} \right) \bigg|_{x=1}^{x=2} = \frac{1}{2} - \frac{1}{2} = 0
\]

\[
\int_1^2 x^{-\frac{2}{3}} \, dx = \left( \frac{3}{-\frac{2}{3} + 1} \right) \bigg|_{x=1}^{x=2} = \frac{3}{\frac{1}{3}} - \frac{3}{\frac{1}{3}} = 9 - 9 = 0
\]

\[
\int_1^2 x^n \, dx = \frac{1}{n + 1} \bigg|_{x=1}^{x=2} = \frac{1}{n + 1} \bigg|_{x=1}^{x=2} = \frac{1}{n + 1} - \frac{1}{n + 1} = 0
\]

\[
\Rightarrow \int_1^2 \left( \sqrt{x} + 3x^5 - \frac{1}{\sqrt[3]{x^2}} \right) \, dx = 1 + 0 + 0 = 1
\]

* Be extra careful with negative signs when evaluating the anti-derivative at the endpoints.

Note that FTC gives a - sign that should be considered in algebra.
Question: What if $n = -1$?

$$\int_1^2 x^{-1} \, dx = \int_1^2 \frac{1}{x} \, dx = \ln x \bigg|_{x=1}^{x=2} = \ln 2 - \ln 1 = \ln 2$$

Let's start with power rule: $\int_1^2 x^{-1} \, dx = \frac{1}{x} \bigg|_{x=1}^{x=2} = \frac{1}{0} - \frac{1}{1}$

When the power is ($-1$), we need something different:

What function $F(x)$ has its derivative equal to $x^{-1} = \frac{1}{x}$?

Remember: $F(x) = \ln x \Rightarrow F'(x) = \frac{1}{x}$

So, $\int_1^2 x^{-1} \, dx = \int_1^2 \frac{1}{x} \, dx = \ln x \bigg|_{x=1}^{x=2} = \ln 2 - \ln 1 = \ln 2$

Example:

Checker Q: $\int_0^1 e^x \, dx = e^x \bigg|_0^1 = e^1 - e^0 = e - 1$

A. 1
B. e
C. $e - 1$
D. 0
E. No idea

For what $F(x)$ we have $(F(x))' = e^x$? $\Rightarrow F(x) = e^x$

What about $\int_0^1 e^{-x} \, dx = -e^{-x} \bigg|_{x=0}^{x=1}$?

Find $F(x)$ such that $F'(x) = e^{-x}$.

Is $F(x) = e^{-x}$ OK? Let's check: $F(x) = (e^{-x})' = e^{-x} \cdot -1 = -e^{-x}$

Multiply by $-1$ and it'll work:

$F(x) = -e^{-x} \Rightarrow F(x) = -e^{-x} \cdot -1 = e^{-x}$
Example: \[ \int_0^\pi \sin x \, dx = \]

What to choose for \( F(x) \) so that \( F'(x) = \sin x \)?

Does \( F(x) = \cos x \) work? Check: \( F'(x) = (\cos x)' = -\sin x \)

So choose \( F(x) = -\cos x \) then \( F'(x) = -(-\sin x) = \sin x \)

\[ \Rightarrow \int_0^\pi \sin x \, dx = -\cos x \bigg|_0^\pi \]

\[ = (-\cos \pi) - (-\cos 0) = (-(-1)) - (-1) \]

\[ = 1 + 1 = 2 \]

Graphically, \( \int_0^\pi \sin x \, dx \) means:

So \( A = 2 \)

What about

\[ \int_0^{2\pi} \sin x \, dx = 0 \]

By formula:

\[ \int_0^{2\pi} \sin x \, dx = -\cos x \bigg|_0^{2\pi} \]

\[ = (-\cos 2\pi) - (-\cos 0) \]

\[ = 1 + 1 = 0 \]