Indefinite integral

Recap:
To find \( \int_0^2 2x \, dx \), we can:

1) Use the definition (the hard way!):
Choose left/right Riemann sum
\[
\int_a^b f(x) \, dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i) \Delta x
\]

2) Find the area under the graph \( y = f(x) \):
\[
\int_0^2 2x \, dx = \text{Area of triangle}
\]
\[
= \frac{4 \times 2^2}{2} = 4
\]

3) Use the Fundamental Theorem of Calculus (FTC):
FTC roughly says, integration and differentiation are opposite each other!
To use FTC:

Find a function, \( F(x) \), such that \( F'(x) = f(x) \).

We call \( F(x) \) an anti-derivative of \( f(x) \).

Then FTC says:

\[
\int_a^b f(x) \, dx = \int_a^b F'(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a)
\]

In our example, \( f(x) = 2x \).

Take \( F(x) = x^2 \), then we see \( F'(x) = (x^2)' = 2x = f(x) \).

So FTC implies:

\[
\int_0^2 2x \, dx = \int_0^2 (x^2)' \, dx = (2)^2 - (0)^2 = 4
\]

But is \( x^2 \) the only anti-derivative of \( 2x \)?

How about \( x^2 + 8 \)? Check: \( (x^2 + 8)' = (x^2)' + (8)' = 2x \) \( \checkmark \)

How about \( x^2 + x^2 \)? Check: \( (x^2 + x^2)' = 2x \) \( \checkmark \)
The general anti-derivative for \( f(x) = 2x \) is given by

\[
F(x) = x^2 + C \quad \text{where } C \text{ is some constant}
\]

Indefinite integral \( \rightarrow \) The general anti-derivative

Indefinite integral: \( \int 2x \, dx = x^2 + C \) for \( C \) some constant

Definite integral: \( \int_2^2 2x \, dx = 4 \)

Another example:

\[
\int x^2 \, dx = \frac{1}{3} x^3 + C \quad \text{where } C \text{ is a constant}
\]

Check: \( \left( \frac{1}{3} x^3 + C \right)' = \frac{1}{3} (3x^2) + 0 = x^2 \checkmark \)

Also

\[
\int e^x \, dx = e^x + C \quad \text{for some } C \text{ constant}
\]

Check: \( (e^x + C)' = (e^x)' + (C)' = e^x + 0 = e^x \)
Does this constant mess up "using FTC to find definite integrals"?  

\[ \int_0^2 2x \, dx = \int_0^2 (x^2 + 4) \, dx \]  

FTC \left\{ \begin{array}{c} x^2 + 4 \bigg|_0^2 = (2^2 + 4) - (0^2 + 4) \\ = 4 + 4 - 0 - 4 = 4 \end{array} \right. \]

No matter \( C \) is, the answer is always the same!  
So, FTC works for any anti-derivative.  

Important remark:  
When using FTC to solve definite integrals, you can take any anti-derivative (usually \( C = 0 \)).  
But be careful, when solving an indefinite integral your answer must have constant \( C \).
General anti-derivatives for some elementary functions: \( n > -1 \), \( n \) is a rational number

\[
\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \quad \text{where } C \text{ is a constant}
\]

\[
\int e^x \, dx = e^x + C
\]

\[
\int \sin x \, dx = -\cos x + C
\]

\[
\int \cos x \, dx = \sin x + C
\]

\[
\int \frac{1}{x} \, dx = \ln |x| + C \quad \text{absolute value as } \ln \text{ only takes positive values}
\]

Also, as before:

\[
\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx
\]

and

\[
\int k \cdot f(x) \, dx = k \int f(x) \, dx \quad \text{where } k \text{ is a constant.}
\]
Example 1:
Find the general anti-derivative of
\[ f(x) = 9e^x + \frac{5}{\sqrt{x}} - 2\cos x + 5 \]

\[ \int f(x) \, dx = \int 9e^x + \frac{5}{\sqrt{x}} - 2\cos x + 5 \, dx \]

\[ = \int 9e^x \, dx + \int \frac{5}{x^{\frac{1}{2}}} \, dx + \int -2\cos x \, dx + \int 5 \, dx \]

\[ = 9e^x + 5 \int x^{-\frac{1}{2}} \, dx - 2 \int \cos x \, dx + 5 \int x^0 \, dx \]

one Constant is enough!

\[ = 9(e^x + C) + 5 \left( \frac{1}{-\frac{1}{2} + 1} x^{-\frac{1}{2} + 1} \right) + \]

\[ -2 \left( \sin x \right) + 5 \left( \frac{1}{0 + 1} x^1 \right) \]

\[ = 9e^x + 9C + 5 \left( 2 x^{\frac{1}{2}} \right) - 2\sin x + 5x \]

\[ = 9e^x + 10x^{\frac{1}{2}} - 2\sin x + 5x + D \]

where \( D \) is a constant.
Example 2:

Use your answer to example 1 to find

\[ \int_0^\pi 9e^x + \frac{5}{\sqrt{x}} - 2 \cos x + 5 \, dx \]

by FTC

\[ = 9e^x + 10x^{1/2} - 2\sin x + 5x + D \left|_0^\pi \right. \]

\[ = [9e^{\pi} + 10\pi^{1/2} - 2\sin(\pi) + 5(\pi) + D] \]

\[ - [9e^0 + 10\sqrt{0} - 2\sin(0) + 5(0) + D] \]

\[ = 9e^\pi + 10\sqrt{\pi} - 0 + 5\pi + D \]

\[ - [9 + 0 - 0 + 0 + D] \]

\[ = 9e^\pi + 10\sqrt{\pi} + 5\pi - 9 \]
Example 3:

Let \( f(x) = \frac{1}{2} \sin x + \sqrt{3} \cos x \).

Find \( F(x) \) such that \( F(x) = f(x) \) and \( F(0) = 2 \).

\[
\int f(x) \, dx = \int \frac{1}{2} \sin x + \sqrt{3} \cos x \, dx
\]

\[
= \frac{1}{2} \int \sin x \, dx + \sqrt{3} \int \cos x \, dx
\]

\[
= \frac{1}{2} (-\cos x + C) + \sqrt{3} (\sin x) \quad C \text{ is a Constant}
\]

\[
= \frac{1}{2} \cos x + \sqrt{3} \sin x + P \quad \text{where } P \text{ is a Constant}
\]

\[
= F(x)
\]

\[
* F(0) = 2 \quad \int_0^0 \frac{1}{2} (-\cos(0) + \sqrt{3} \sin(0)) + P = -\frac{1}{2} + P = 2
\]

\[
F(0) = -\frac{1}{2} \cos(0) + \sqrt{3} \sin(0) + P = -\frac{1}{2} + P = 2
\]

\[
\Rightarrow P = \frac{5}{2}
\]

\[
F(x) = -\frac{1}{2} \cos x + \sqrt{3} \sin x + \frac{5}{2}
\]