Last Week:

→ **Definite Integral:**

\[
\int_a^b f(x) \, dx = \int_a^b F'(x) \, dx = F(b) - F(a) \quad \text{This is a number.}
\]

Find an anti-derivative

\[ F(x) \] such that

\[ F'(x) = f(x) \]

\[ \text{FTC: } \int \text{ and } \int \text{ cancel each other.} \]

→ **Indefinite Integral:**

\[
\int f(x) \, dx = \frac{F(x) + C}{2} \quad \text{A function } f(x) \text{ can have many \text{anti-derivatives } F(x), \text{ but they're all different by a constant term.}}
\]

* For any constant \( C \):

\[
(F(x) + C)' = F'(x) = f(x)
\]

**Clicker Q.** Find the general anti-derivative of

\[ f(x) = \frac{x}{2} - e^x + \sin x - \frac{5}{\sqrt{x}} + 2 \]

\[ \text{term by term: } \frac{x}{2} \rightarrow \frac{1}{2} \cdot \frac{1}{1+1} x^{1+1} = \frac{1}{4} x^2 \quad , \quad e^x \rightarrow e^x \quad , \quad \sin x \rightarrow -\cos x \]

\[ \frac{5}{\sqrt{x}} \rightarrow 5 \cdot \frac{1}{-\frac{1}{2}+1} = 5 \cdot \frac{1}{\frac{1}{2}} = 10 \sqrt{x} \]

A. \( F(x) = \frac{x^2}{4} - e^x - \cos x - 10\sqrt{x} + 2 + C \)

B. \( F(x) = \frac{x^2}{4} - e^x + \cos x + 10\sqrt{x} + 2x + C \)

\[ \text{C. (Correct)} \quad F(x) = \frac{x^2}{4} - e^x - \cos x - 10\sqrt{x} + 2x + C \]

D. \( F(x) = 2x^2 - e^x - \cos x - 10\sqrt{x} + C + \frac{2x}{2} \)
**Question:** How should we compute \[ \int (2x+1)^2 \, dx \]

We want to find the anti-derivative of \((2x+1)^2\) by using integration technique.

From the question above, we know that the answer should be \( F(x) = \frac{1}{6} (2x+1)^3 + C \) now, let's see how we find it.

**Easy case:** \[ \int x^2 \, dx = \frac{1}{3} x^3 + C \]

Let's write the integral in a form whose integral is easy to find:

Take \(2x+1 = u\) then integral becomes \[ \int u^2 \, dx \]

The variables in the integral must be all the same, so we need to change \(dx\) to \(du\):

we have \(2x+1 = u\)

Derive both sides with respect to \(x\): \[ \frac{d}{dx} (2x+1) = \frac{du}{dx} \Rightarrow 2 = \frac{du}{dx} \Rightarrow dx = \frac{du}{2} \]

Now integral becomes: \[ \int u^2 \cdot \frac{du}{2} = \frac{1}{2} \int u^2 \, du \]

we know how to compute this:

\[ = \frac{1}{2} \cdot \frac{1}{3} u^3 + C \]

\[ = \frac{1}{6} u^3 + C = \frac{1}{6} (2x+1)^3 + C \]

**How to verify?** Check if \( F'(x) = f(x) \):

\[ F(x) = \frac{1}{6} (2x+1)^3 + C \]

**Chain rule:** \[ F'(x) = \frac{1}{6} \cdot 3 \cdot (2x+1)^2 \cdot 2 = \frac{6}{6} (2x+1)^2 = (2x+1)^2 \]

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**Easy case:** \[ \int x^2 \, dx = \frac{1}{3} x^3 + C \]

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\[ = \frac{1}{6} u^3 + C = \frac{1}{6} (2x+1)^3 + C \]
Example: \[ \int e^{5x} \, dx = \]

We know how to compute \( \int e^x \, dx \), so let's rewrite the integral with a u-substitution:

take \( 5x = u \) then \( \frac{d}{dx}(5x) = \frac{du}{dx} \Rightarrow 5 \, dx = du \Rightarrow dx = \frac{du}{5} \)

\[
\int e^{5x} \, dx = \int e^u \, dx = \int e^u \cdot \frac{du}{5} = \frac{1}{5} \int e^u \, du = \frac{1}{5} e^u + C = \frac{1}{5} e^{5x} + C
\]

* This technique in integration is called **substitution**. We need to know which piece should be substituted by \( u \) so that when we replace \( dx \) and simplify, we are left with a relatively easier integral only in terms of \( u \).

Example: \[ \int 3x \cos(x^2) \, dx = \]

→ What substitution would you use ...

3 choices:

A. \( x^2 = u \)

B. \( \cos(x^2) = u \)

C. \( 3x = u \)

1st choice: \( x^2 = u \) \( \Rightarrow \) \( 2x \, dx = \frac{du}{dx} \Rightarrow 2 \, dx = du \Rightarrow dx = \frac{du}{2x} \)

Rewrite the integral:

\[
\int 3x \cos(x^2) \, dx = \int 3x \cdot \cos(u) \cdot \frac{du}{2x} = \frac{3}{2} \int \cos u \, du \quad \text{→ easy to evaluate and only in term of } u
\]

\[= \frac{3}{2} \sin u + C = \frac{3}{2} \sin(x^2) + C \]
What about other choices:

B. If $\cos(x^2) = u \Rightarrow \frac{d}{dx} (\cos(x^2)) = \frac{du}{dx}$

Chain rule $\Rightarrow -\sin(x^2) \cdot 2x = \frac{du}{dx}$

Solve for $\Rightarrow -\sin(x^2) \cdot 2x \cdot dx = du \Rightarrow dx = -\frac{du}{\sin(x^2) \cdot 2x}$

Rewrite the integral: $\int 3x \cos(x^2) \, dx = \int 3x \cdot u \cdot -\frac{du}{\sin(x^2) \cdot 2x}$

$= -\frac{3}{2} \int \frac{u}{\sin(x^2)} \, du \Rightarrow$ Not an improvement from the original integral, still involves

C. If $3x = u \Rightarrow 3 \, dx = du \Rightarrow dx = \frac{du}{3}$

$\int 3x \cos(x^2) \, dx = \int u \cos(x^2) \frac{du}{3}$

$= \frac{1}{3} \int u \cos(x^2) \, du \Rightarrow$ Not an easier integral

Conclusion: There is no fixed rule that tells us how to substitute. By practicing, we'll learn how to foresee one step ahead to know what substitution will eventually simplify all x's and we get an integral only in terms of u.

→ Hint: Choose the substitution such that its derivative is somehow existing in the integral.
1) Evaluate the following indefinite integrals.

\[ a) \int \frac{x^2}{(1+x^3)^2} \, dx \]
\[ b) \int \frac{\ln x}{x} \, dx \]
\[ c) \int \sqrt{4-x} \, dx \]
\[ d) \int (2x+5)(x^2+5x)^7 \, dx \]
\[ e) \int \tan x \, dx \]
\[ f) \int \frac{\sin^2 x \cos x}{\sin(5x)} \, dx \]
\[ g) \int \frac{\cos(5x)}{e^{\sin(5x)}} \, dx \]
\[ h) \int x^2 e^{-4x^3} \, dx \]

2) Evaluate the following definite integrals.

\[ a) \int_{-1}^{1} \frac{x+1}{(x^2+2x+2)^3} \, dx \]
\[ b) \int_{0}^{\pi} \cos x \cdot \sqrt{\sin x} \, dx \]
\[ c) \int_{-1}^{1} x^2 \sqrt{x^3+1} \, dx \]
\[ d) \int_{1}^{4} \frac{1}{\sqrt{x} (\sqrt{x} +1)^2} \, dx \]