Integration by Parts

\[ \int u \, dv = uv - \int v \, du \rightarrow \text{anti-product rule} \]

You are given an integral, you choose \( u \) and \( dv \) and you apply the above formula.

Example 1:

\[ \int xe^x \, dx = \int u \, dv = uv - \int v \, du \]

Choose \( u \) and \( dv \):

\[ x = u \quad e^x \, dx = dv \]

\[ 1 \cdot dx = du \quad e^x = v \quad \text{(anti-derive)} \]

\[ = xe^x - \int e^x \, dx \]

\[ = xe^x - e^x + C \]

Let's check if we did it correctly:

\[ (xe^x - e^x + C)' = e^x + xe^x - e^x = xe^x \]

Question: Does it matter how we choose \( u \) and \( dv \)?

What if I choose \( u = e^x \) and \( x \, dx = dv \)?

\[ u = e^x \quad x' \, dx = dv \]

\[ du = e^x \, dx \quad \frac{1}{1+1} \cdot x^{1+1} = v \]

\[ \frac{1}{2} x^2 = v \]

\[ \times \text{ NOT a good choice} \]

\[ e^x \cdot \frac{1}{2} x^2 - \int \frac{1}{2} x^2 e^x \, dx \]

even harder than the original one.
Usually our choice for “U” is prioritized by the following list:

(1) Logarithmic functions
(2) Algebraic functions such as polynomials
(3) Trig functions
(4) Exp functions.

Last Example: \( \int x \cdot \ln x \, dx \)

\( e^x \) → exp function → \( x = u \)

\( \ln x = u \quad x \, dx = dv \)

\( \frac{1}{x} \, dx = du \quad \frac{1}{2} x^2 = v \)

\( \int x \cdot \ln x \, dx = \int u \, dv = uv - \int v \, du = x \ln x - \frac{1}{2} x^2 - \int \frac{1}{x} \, dx \)

\( = \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx \)

\( = \frac{1}{2} x^2 \ln x - \frac{1}{2} \left( \frac{1}{2} x^2 \right) + C \)

Remark: Note that IBP is different from substitution, even if we use “U”. In substitution, we transform the integral \( \int f(x) \, dx \) to a new integral \( \int f(u) \, du \), so the variable is changing from \( x \) to \( u \). However, in IBP we choose “u” and “dv” so that we can use the formula and simplify the integral, but the variable in the integral is still the original variable: \( x \).
Clicker Q: \[ \int x \sin x \, dx \]

How would you choose u and dv?

A. \( u = x \), \( dv = \sin x \, dx \)

B. \( u = \sin x \), \( dv = x \, dx \)

\[ u = x \quad dv = \sin x \, dx \]

\[ du = dx \quad v = -\cos x \]

\[ \int x \sin x \, dx = \int u \, dv = uv - \int v \, du \]

\[ = -x \cos x - \int -\cos x \, dx \]

\[ = -x \cos x + \int \cos x \, dx \]

\[ = -x \cos x + \sin x + C \]

**Question:** Find a particular function \( F(x) \) whose derivative \( F'(x) = x \sin x \) and \( F\left(\frac{\pi}{2}\right) = 3 \).
Definite integral with IBP keep the integral bounds all along:

\[ \int_{a}^{b} u \, dv = uv \bigg|_{a}^{b} - \int_{a}^{b} v \, du \]

\[ \text{Ex 4: } \int_{0}^{\frac{\pi}{2}} x^2 \sin x \, dx = \pi - 2 \]

\[ u = x^2 \quad \sin x \, dx = dv \]

\[ du = 2x \, dx \quad -\cos x = v \]

\[ \int_{0}^{\frac{\pi}{2}} x^2 \sin x \, dx = \int_{0}^{\frac{\pi}{2}} u \, dv = uv \bigg|_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} v \, du \]

\[ = -x^2 \cos x \bigg|_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} -\cos x \cdot 2x \, dx \]

- \[ -x^2 \cos x \bigg|_{0}^{\frac{\pi}{2}} = -\left( \frac{\pi}{2} \right)^2 \cos \frac{\pi}{2} - \left( -0^3 \cos 0 \right) = 0 \]

- \[ 2\int_{0}^{\frac{\pi}{2}} x \cos x \, dx = 2 \left[ x \sin x \bigg|_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} \sin x \, dx \right] \]

Another step with IBP

\[ x = u \quad \cos x \, dx = dv \]

\[ dx = du \quad \sin x = v \]

- \[ x \sin x \bigg|_{0}^{\frac{\pi}{2}} = \frac{\pi}{2} \sin \frac{\pi}{2} - 0 \sin 0 = \frac{\pi}{2} \]

- \[ \int_{0}^{\frac{\pi}{2}} \sin x \, dx = -\cos x \bigg|_{0}^{\frac{\pi}{2}} = -\cos \frac{\pi}{2} - (\cos 0) = 1 \]
Practice Problems:

a) \[ \int 3x e^{-x} \, dx \]

b) \[ \int \frac{\ln x}{x^2} \, dx \]

c) \[ \int x^2 \cos x \, dx \]

d) \[ \int \ln x \cdot 1 \, dx \]

e) \[ \int e^x \sin x \, dx \]

f) \[ \int -x^3 e^{x^2} \, dx \]

g) \[ \int \sin x \cos x \, dx \]

h) \[ \int \cos^2 x \cdot \ln(\sin x) \, dx \]

Definite Integrals

a) \[ \int_{-1}^{1} (2x + 3)^3 (-x+2) \, dx \]

b) \[ \int_{0}^{\frac{\pi}{2}} (-6x + 4) \cos x \, dx \]

c) \[ \int_{1}^{e^3} x^7 \ln x \, dx \]

d) \[ \int_{\frac{-\pi}{6}}^{\frac{\pi}{3}} 2x \cos(3x + \pi) \, dx \]
Practice Problems:

(d) \[ \int \ln x \, dx = uv - \int v \, du \]
\[ u = \ln x \quad 1 \cdot dx = dv \]
\[ du = \frac{1}{x} \, dx \quad x = v \]

\[ = x \ln x - \int x \cdot \frac{1}{x} \, dx \]
\[ = x \ln x - \int 1 \, dx \]
\[ = x \ln x - x + C \]

(e) \[ \int e^x \sin x \, dx \]
\[ u = \sin x \quad e^x \, dx = dv \]

Will complete it next class.