1. A 20m tree has has been bent in a storm and makes an angle of 60° with the ground. Some sap is moving down the tree moving at speed 2 m/min. How fast is the distance from the sap to the ground decreasing when the sap is half way down the tree?

Solution: Call the distance from the sap to the ground $x$ and call the distance along the tree from the sap to the ground $y$. We know that $dy/dt$ is decreasing at a rate of 2 m/min and would like to find $dx/dt$.

![Diagram of triangle with angle $\pi/3$]

Using the above figure and trigonometry we see that

$$\sin \left( \frac{\pi}{3} \right) = \frac{x}{y}.$$ 

We know that $\sin \pi/3 = \sqrt{3}/2$ and we rearrange the equation to read

$$\frac{\sqrt{3}}{2} y = x.$$ 

We now differentiate both sides with respect to $t$ to achieve

$$\frac{\sqrt{3}}{2} \frac{dy}{dt} = \frac{dx}{dt}.$$ 

With some substitution we find the desired value.

$$\frac{dx}{dt} = \frac{\sqrt{3}}{2} (-2 \text{ m/min}) = -\sqrt{3} \text{ m/min}.$$ 

In this way we now know that $dx/dt$ is decreasing at a rate of $\sqrt{3}$ m/min.

Solution 2: We can also use quotient rule after

$$\sin \left( \frac{\pi}{3} \right) = \frac{x}{y}.$$
Let \( x' = dx/dt \) and \( y' = dy/dt \) then

\[
0 = \frac{x'y - xy'}{y^2}
\]
or rather

\[
x'y = xy' \quad x' = \frac{x}{y}y'.
\]

We now note that \( x/y = \sin \pi/3 \) and so

\[
x' = \sin \left( \frac{\pi}{3} \right) y' = \frac{\sqrt{3}}{2}(-2) = -\sqrt{3}.
\]

**Solution3:** We can try to use the Pythagorean Theorem but we have a hard time with it and end up having the use more or less the same trig anyway. Call the remaining side length \( z \) so then

\[
x^2 + z^2 = y^2.
\]

We don’t like having \( z \) since we know nothing about \( dz/dt \). Let’s get rid of it now in favour of \( y \) and \( x \). Using some trig we know that \( \cos \pi/3 = z/y \) so \( z = y \cos \pi/3 \). There follows

\[
x^2 + \cos^2(\pi/3)y^2 = y^2
\]

and

\[
x^2 = (1 - \cos^2(\pi/3)) y^2
\]

\[
= \sin^2(\pi/3)y^2
\]

with our favourite trig identity. Taking now the derivative of both sides in \( t \) we see

\[
2xx' = 2\sin^2(\pi/3)yy'
\]

so

\[
x' = \sin^2(\pi/3)\frac{y}{x}y'.
\]

Recall that \( y/x = 1/\sin(\pi/3) \) to finally achieve

\[
x' = \frac{\sin^2(\pi/3)}{\sin(\pi/3)}y'
\]

\[
x' = \sin(\pi/3)y'
\]
as before.