Series solutions of 2nd order variable-coefficient Linear ODEs:

Consider the homogeneous ODE:

\[ P(x) y'' + Q(x) y' + R(x) y = 0 \]  \hspace{1cm} (1)

Divide by \( P(x) \):

\[ L y = y''(x) + p(x) y' + q(x) y = 0 \]

where \( p(x) = \frac{Q(x)}{P(x)} \), \( q(x) = \frac{R(x)}{P(x)} \)

If \( p(x) \) and \( q(x) \) are analytic functions at \( x = x_0 \), i.e. they are infinitely differentiable, and we can find their Taylor expansions:

\[ p(x) = p(x_0) + p'(x_0) (x-x_0) + \cdots + \frac{p^{(n)}(x_0)}{n!} (x-x_0)^n + \cdots = \sum_{n=0}^{\infty} \frac{p_n(x-x_0)^n}{n!} \]

\[ q(x) = q(x_0) + q'(x_0) (x-x_0) + \cdots + \frac{q^{(n)}(x_0)}{n!} (x-x_0)^n + \cdots = \sum_{n=0}^{\infty} \frac{q_n(x-x_0)^n}{n!} \]

Then the expansion point is an ordinary point of the ODE in (1).
For ordinary points it is possible to find a solution using the power series expansion:

\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad x_0: \text{an ordinary point} \]

\[ = a_0 y_1(x) + a_1 y_2(x) \]

\[ y_1, y_2: \text{linearly independent.} \]

- If \( p(x) \) and \( q(x) \) are not analytic at \( x_0 \), e.g. if \( p(x_0) = 0, q(x_0) \neq 0, R(x_0) \neq 0 \), then \( x_0 \) is a singular point.

- If \( x_0 \) is an ordinary point, then the radius of convergence of the solution is at least as large as the distance from the expansion point \( x_0 \) to the nearest singular point.
- Singular points are divided to two classes:
  - regular singular points: Frobenius series solution
  - irregular singular points: beyond this course.

Example of a regular singular point:

Cauchy-Euler equation:

\[(x - x_0)^2 y'' + \alpha(x - x_0)y' + \beta y = 0\]

\[x = x_0\] is a singular point of the Cauchy-Euler equation. So power series solution fails at \[x = x_0\]. This point is the simplest example of an irregular singular point.

\[y(x) = (x - x_0)^r\], the idea is this solution is a simple case of the more general solution:
\( y(x) = (x-x_0)^r \left[ a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \right] + \cdot \cdot \cdot \)

\( = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n \)

where for the cauchy-euler eqn.

\( a_1 = a_2 = \cdots = 0 \), \( a_0 \neq 0 \).

Regular and irregular singular points:

Consider the variable-coefficient linear 2nd order ODE:

\( P(x) y'' + Q(x) y' + R(x) y = 0 \)

Consider the case where \( P(x) \), \( Q(x) \) and \( R(x) \) are polynomials, \( P(x_0) = 0 \) and at least one of \( Q(x_0) \) and \( R(x_0) \) is \( \neq 0 \).

Then \( x_0 \) is a singular point of the ODE.
Point $x_0$ is a regular singular point if:

$$\lim_{x \to x_0} \left( x - x_0 \right) \frac{Q(x)}{P(x)}$$ is finite, and

$$\lim_{x \to x_0} \left( x - x_0 \right)^2 \frac{R(x)}{P(x)}$$ is also finite.

This means that the singularity in $\frac{P(x)}{Q(x)}$ is not worse than $(x - x_0)^{-1}$ and the singularity in $\frac{R(x)}{Q(x)}$ is not worse than $(x - x_0)^{-2}$, or singularities in $\frac{Q}{P}$ and $\frac{R}{P}$ are "weak singularities".

Any singular point that is not regular, is called irregular singular point.

Example:
Classify the singular points of the Legendre equation:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$
Singular points:

\[ 1 - x^2 = 0 \quad \rightarrow \quad x = 1 \quad \text{and} \quad x = -1 \]

\[
\lim_{x \to 1} \frac{-2x}{1-x^2} = \lim_{x \to 1} \frac{2x}{1+x} = 1
\]

\[
\lim_{x \to 1} \frac{x(x+1)}{(1-x)} = \lim_{x \to 1} \frac{x-1}{(1+x)} = 0
\]

These limits are finite, so \( x = 1 \) is a regular singular point.

\[
\lim_{x \to -1} \frac{-2x}{1-x^2} = \lim_{x \to -1} \frac{-2x}{1-x} = 1
\]

\[
\lim_{x \to -1} \frac{x(x+1)}{1-x^2} = \lim_{x \to -1} \frac{x+1}{(1+x)} = 0
\]

\( x = -1 \) is also a regular singular point.

Example:

\[ Ly = 2x^2 y'' - xy' + (1-x)y = 0 \]

\[ P(x) = 2x^2, \quad Q(x) = -x, \quad R(x) = 1-x \]

\( x \neq 0 \) : ordinary point

\( x = 0 \) : singular point
\[
\lim_{x \to 0} \frac{Q(x)}{P(x)} = \lim_{x \to 0} x \cdot \frac{(-x)}{2x^2} = -\frac{1}{2}
\]

\[
\lim_{x \to 0} \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 \frac{(1-x)}{2x^2} = \frac{1}{2}
\]

A regular singular point.

Example:

\[
x y'' - y = 0 \quad P(x) = x^3, \quad Q(x) = 0, \quad R(x) = -1
\]

\[x \neq 0 : \text{ordinary point}\]

\[x = 0 : \text{singular point}\]

\[
\lim_{x \to 0} \frac{x^2 R(x)}{P(x)} = \lim_{x \to 0} \frac{x^2 (-1)}{x^3} = 0
\]

\[x = 0 \text{ is an irregular singular point.}\]
Series solution near a regular singular point:

\[ P(x) y'' + Q(x) y' + R(x) y = 0 \quad (*) \]

If \( x_0 \) is a regular singular point, then

\[ \lim_{x \to x_0} \frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \to x_0} \frac{(x-x_0)^2 R(x)}{P(x)} \]

are finite and we can write:

\[ P(x) = (x-x_0) \frac{Q(x)}{P(x)} = P_0 + P_1(x-x_0) + P_2(x-x_0)^2 + \ldots \]

\[ q(x) = (x-x_0)^2 \frac{R(x)}{P(x)} = q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \ldots \]

In other words, \( (x-x_0) \frac{Q}{P} \) and \( (x-x_0)^2 \frac{R}{P} \)

are analytic functions.

Divide equation (*) by \( P(x) \) and multiply it by \( (x-x_0)^2 \):

\[ (x-x_0)^2 y'' + (x-x_0) P(x) y' + q(x) y = 0 \]

\((***)\)
Equation (**) is similar to a Cauchy-Euler equation, except that constants \(\alpha, \beta\) are replaced by \(p(x)\) and \(q(x)\).

Now, substitute for \(p(x)\) and \(q(x)\):

\[
(x - x_0)^2 y'' + (x - x_0) \left[ p_0 + p_1 (x - x_0) + p_2 (x - x_0)^2 + \ldots \right] y' + \left[ q_0 + q_1 (x - x_0) + q_2 (x - x_0)^2 + \ldots \right] y = 0
\]

If \(x\) is close to \(x_0\), then \((x - x_0)\) is small, so:

\(p_1 (x - x_0) < p_0\) , \(p_2 (x - x_0)^2 < p_0\) , \ldots

Let's move all the small terms to the RHS:

\[
(x - x_0)^2 y'' + (x - x_0) P_0 \ y' + q_0 \ y = P_1 (x - x_0) y' + P_2 (x - x_0)^2 y' + q_1 (x - x_0) y + q_2 (x - x_0)^2 y + \ldots
\]

Now, as a first step we can neglect all the small terms and solve:
\[(x-x_0)^2 y'' + (x-x_0) P_0 y' + q_0 y = 0\]

\[P_0 = \lim_{x \to x_0} (x-x_0) \frac{Q(x)}{P(x)} \quad \text{and} \quad q_0 = \lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)}\]

This is a Cauchy-Euler equation and has the solution of the form: \[y(x) = (x-x_0)^r\]

Now, because of the presence all the terms on the RHS of (**) we need to modify \(y_0\) by adding a particular solution to it.

It is plausible to seek a solution of the form:

\[y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad (\alpha \neq 0)\]

This part catches the singularity near \(x_0\)

This is a Frobenius series.

This part is the correction.