Consider the ODE:

\[ p(x) y'' + q(x) y' + r(x) y = 0 \]

If \( x_0 \) is a regular singular point of the ODE, the Frobenius solution near this point has the form:

\[ y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

For the solution to be complete, we need to:

1. Find the values of \( r \)

2. Find the recursion relation for \( n \)

3. Find the radius of convergence of \( \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} \)
Example:

\[ Ly = 2x^2 y'' - xy' + (1-x) y = 0 \quad (1) \]

\( x_0 = 0 \) is a singular point.

\[ \lim_{x \to 0} \frac{-x}{2x^2} = -\frac{1}{2} = p_0 \]

\[ \lim_{x \to 0} \frac{(1-x)}{2x^2} = \frac{1}{2} = q_0 \]

\( x_0 = 0 \) is a regular singular point.

First method: Solve the approximate equation:

(a sequence of equations)

As \( x \to 0 \) near the singular point

\( 1-x \to 1 \) or: \( y \gg xy \)

So you can approximately say:

\[ L_0 y = 2x^2 y'' - xy' + y = xy = 0 \quad (2) \]

which is a cauchy-Euler eqn.

\[ y = x^r \rightarrow \begin{array}{l}
2r(r-1) - r + 1
\end{array} \]

\[ 2r^2 - 3r + 1 = 0 \rightarrow r = \frac{1}{2}, 1 \]

\[ y_0 = c_1 x^{\frac{1}{2}} + c_2 x \]
We need to correct this solution to take into account the forcing on the RHS of (2):

\[ L_1 y = 2x^2 y'' - xy' + y = c_1 x^{3/2} + c_2 x^2 \]  \hspace{1cm} (3)

guess a particular solution:

\[ \begin{align*}
  y_1 p &= A_0 x^{3/2} + B_0 x^2 \\
  y_1' p &= 3/2 A_0 x^{1/2} + 2B_0 x \\
  y_1'' p &= 3/4 A_0 x^{-1/2} + 2B_0 \\
\end{align*} \]

Substitute in (3):

\[ \begin{align*}
  3/2 A_0 x^{3/2} + 4B_0 x^2 - \frac{3}{2} A_0 x^{3/2} - 2B_0 x^2 + A_0 x^{3/2} + B_0 x^2 &= c_1 x^{3/2} + c_2 x^2 \\
\end{align*} \]

Collect all coefficients of \( x^{3/2} \) and \( x^2 \) and make them equal on both sides:

\[ \begin{align*}
  3/2 A_0 - 3/2 A_0 + A_0 &= c_1 \\
  4B_0 - 2B_0 + B_0 &= c_2 \\
\end{align*} \]

\[ \begin{align*}
  A_0 &= C_1 \\
  B_0 &= \frac{C_2}{3} \\
\end{align*} \]
So: \( y_1(x) = C_1 x^{1/2} (1+x) + C x (1 + x^{1/3}) \)

You can substitute \( y_1(x) \) in (2) and get the higher order terms in power series.

**Method 2**: We can also use the Frobenius solution from the start:

\[
2 x^2 y'' - xy' + (1-x) y = 0, \quad P(x) = 2x^2, \quad Q(x) = -x, \quad R(x) = 1-x
\]

\[
P_0 = \lim_{x \to 0} x \frac{Q(x)}{P(x)} = -\frac{1}{2}
\]

\[
q_0 = \lim_{x \to 0} x \frac{R(x)}{P(x)} = \frac{1}{2}
\]

The corresponding Cauchy-Euler (or equidimensional) equation is:

\[
x^2 y'' - \frac{1}{2} xy' + \frac{1}{2} y = 0 \quad \rightarrow \quad \text{which has solutions:}
\]

\[x^1, x^{1/2}\]
The Frobenius Solution is:

\[ y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \]

\[ y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \]

\[ y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \]

Substitute in the ODE:

\[ 2x^2 y'' - xy' + (1-x) y = 0 \]

\[ 2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} \]

\[ - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^n = 0 \]

Shift the index:

\[ n+r+1 = m+r \]

\[ n+1 = m \]

\[ n = m-1 \]

\[ n = 0 \rightarrow m = 1 \]

Also take the \( n=0 \) terms out of the first three series:
\[
2a_0 r(r-1)x^r - a_0 rx^r + a_0 x^r + \sum_{n=1}^{\infty} \left[ 2a_n (n+r)(n+r-1) - a_n (n+r) + a_n - a_{n-1} \right] x^{n+r} = 0
\]

\[
x^r \int a_0 (2r(r-1)-r+1)x^r = 0
\]

\[
2r(r-1) - r + 1 = 0
\]

The indicial equation

\[
\rightarrow 2r^2 - 3r + 1 = 0
\]

\[
r = 1, \quad \frac{1}{2}
\]

\[
x^{n+r}, \quad n \geq 1
\]

\[
a_n (2(n+r)(n+r-1) - (n+r)+1) = a_{n-1}
\]

\[
a_n = \frac{a_{n-1}}{(n+r)(2(n+r)-3) + 1}
\]

The recursion relation

We have to find the recursion for \( r_1 = 1 \), and \( r_2 = \frac{1}{2} \).
\[ r_1 = \frac{1}{2} : \quad a_n = \frac{a_{n-1}}{(n+\frac{1}{2})(2n+1-3)+1} \]

\[ = \frac{a_{n-1}}{2(n+\frac{1}{2})(n-1)+1} \]

\[ = \frac{a_{n-1}}{2\left(n^2 - \frac{1}{2}n - \frac{1}{2}\right) + 1} \]

\[ = \frac{a_{n-1}}{2n^2 - n - 1 + 1} = \frac{a_{n-1}}{n(2n-1)} \]

\[ n=1 : \quad a_1 = \frac{a_0}{1.1} \quad , \quad n=2 : \quad a_2 = \frac{a_1}{2.3} = \frac{a_0}{2.3} = \frac{a_0}{6} \]

\[ n=3 : \quad a_3 = \frac{a_2}{3.5} = \frac{a_0}{15.6} = \frac{a_0}{90} \]

\[ y_1(x) = a_0 x^{1/2} \left[ 1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \ldots \right] \]

\[ r_2 = 1 : \quad a_n = \frac{a_{n-1}}{(n+1)(2(n+1)-3)+1} \]

\[ = \frac{a_{n-1}}{(n+1)(2n-1)+1} = \frac{a_{n-1}}{2n^2+n-1+1} \]
\[ a_n = \frac{a_{n-1}}{n(2n+1)}, \text{ for } n \geq 1 \]

\[ n = 1: \quad a_1 = \frac{a_0}{1.3}, \quad n = 2: \quad a_2 = \frac{a_1}{2.5} = \frac{a_0}{30} \]

\[ n = 3: \quad a_3 = \frac{a_2}{3.7} = \frac{a_0}{630} \]

\[ y_2(x) = a_0 x^3 \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \ldots \right) \]

Note: Compare this to the solution from previous method.

The last step is to find radius of convergence of series:

ratio test:

\[ \lim_{n \to \infty} \left| \frac{a_n x^n}{a_{n-1} x^{n-1}} \right| = \lim_{n \to \infty} \left| x \right| \cdot \left| \frac{1}{(n+r)((2n+r)-3)+1} \right| \]

\[ = 0 \]

So, \( r = \infty \) for all \( x \).
The general solution is:

\[ y(x) = c_1 x^{1/2} \left[ 1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \ldots \right] \]

\[ + c_2 x \left[ 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \ldots \right] \]

Examples of RSP, ESP and radius of convergence:

Example 1:

\[ x(1+x^2)y'' + 2xy' + (x^2 + 1)y = 0 \]

a- Find and classify the singular points.

b- What is the lower bound on the radius of convergence of power series expansion about \( x_0 = 2 \)?

a- Singular points:

\[ P(x) = x(1+x^2), \quad Q(x) = 2x, \quad R(x) = x^2 + 1 \]
\[
\frac{Q}{P} = \frac{2x}{x(1+x^2)} = \frac{2}{1+x^2} \rightarrow x_o = \pm i
\]
singular points

\[
\frac{R}{P} = \frac{x+1}{x(x^2+1)} = \frac{1}{x} \rightarrow x_o = 0 \text{ as a singular point}
\]

Classify the singular points:

\(x_o = 0:\)

\[
\lim_{x \to 0} \frac{Q}{P} x = \lim_{x \to 0} \frac{2x}{1+x^2} = 0 \rightarrow x_o = 0 \text{ is a regular singular point}
\]

\(x_o = i:\)

\[
\lim_{x \to i} \frac{Q}{P} (x-i) = \lim_{x \to i} \frac{2(x-i)}{1+x^2} \overset{\text{L'Hopital's rule}}{=} \lim_{x \to i} \frac{2}{2x} = \frac{1}{i} \text{ finite}
\]

\[
\lim_{x \to i} \frac{R}{P} (x-i)^2 = \lim_{x \to i} \frac{1}{x} (x-i)^2 = 0 \quad x_o = i \text{ is a regular singular point}
\]

\(x_o = -i:\)

\[
\lim_{x \to -i} \frac{Q}{P} (x+i) = \lim_{x \to -i} \frac{2(x+i)}{1+x^2} = \lim_{x \to -i} \frac{2}{2x} = \frac{-1}{i}
\]

\[
\lim_{x \to -i} \frac{R}{P} (x+i)^2 = \lim_{x \to -i} \frac{(x+i)^2}{x} = 0 \quad x_o = -i \text{ a RSP}
\]
radius of convergence near $x_0 = 2$:

The distance from $x_0 = 2$ to the nearest singular point. You have to find this distance on the complex plane:

$$\sqrt{2^2 + 1} = \sqrt{5}$$

$p = 2$ : the lower bound for the radius of convergence.
Example:

\[ \cos x \cdot y'' + y' + \cot(x) \cdot y = 0 \]

Find the singular points of this ODE. Classify them as regular or irregular.

\[ \frac{Q}{P} = \frac{1}{\cos x} \quad \frac{R}{P} = \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} = \frac{1}{\sin x} \]

Singular points:

\[ \cos x = 0 \quad \rightarrow \quad x = (2n+1) \frac{\pi}{2} \]
\[ \sin x = 0 \quad \rightarrow \quad x = m \pi \]

Classification:

\[ \lim_{x \to (2n+1) \frac{\pi}{2}} \frac{x - (2n+1) \frac{\pi}{2}}{\cos x} = \lim_{x \to (2n+1) \frac{\pi}{2}} \frac{1}{-\sin x} = (-1)^{2n+1} < \infty \]

\[ \lim_{x \to (2n+1) \frac{\pi}{2}} \frac{(x - (2n+1) \frac{\pi}{2})^2}{\sin x} = 0 \]

So, \( x = (2n+1) \frac{\pi}{2} \) are regular singular points.
\[
\begin{align*}
\lim_{x \to m\pi} \frac{(x - m\pi)}{\cos x} &= 0 \\
\lim_{x \to m\pi} \frac{(x - m\pi)^2}{\sin x} &= \lim_{x \to m\pi} \frac{2(x - m\pi)}{\cos x} = 0 \\
\end{align*}
\]

\(x = m\pi\) are also RSP.