Bessel’s equation:

Applications:

- PDEs on circular/cylindrical domains

Bessel equation of order $v$:

$$L y = x^2 y'' + xy' + (x^2 - v^2) y = 0 \quad (1)$$

$v$: a constant, $v \in \mathbb{Z} = \{..., -1, 0, 1, 2, ...\}$

$x_0 = 0$ is a regular singular point:

$$\lim_{x \to 0} x^r \cdot x = 1 = p_0, \lim_{x \to 0} \frac{(x^2 - v^2) x^2}{x^2} = -v^2 = q_0$$

So, the indicial equation is:

$$r (r-1) + p_0 r + q_0 = 0$$

or $r (r-1) + r - v^2 = 0 \rightarrow r^2 - v^2 = 0$

$$r = \pm v \quad \text{roots}$$

Let: $y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (a \text{ Frobenius series})$
\[ y' = \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2} \]

Substitute into (1):

\[ Ly = x^2 y'' + xy' + x^2 y - \nu^2 y = 0 \]

\[ Ly = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r)x^n \]

\[ + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^n = 0 \]

↑ shift the index for this series

\[ n + 2 = m, \quad n = m - 2 \]

\[ n = 0 \Rightarrow m = 2 \]

Expand the first two terms out of series 1, 2, 4:

\[ a_0 r (r-1) x^r + a_1 (r+1) r x^{r+1} + a_0 x^r + a_1 (r+1) x^{r+1} - \nu a_0 x^2 - \nu a_1 x^3 \]

\[ + \sum_{n=2}^{\infty} \sum_{n=2}^{\infty} \left[ a_n (n+r)(n+r-1) + a_{n+1} (n+1) + a_{n-2} - \nu^2 a_n \right] x^{n+r} = 0 \]
Set the coefficients of all powers of \( x \) to zero:

(linear independency)

\[
\begin{align*}
\sum_{r} a_{r} r(r-1) + a_{r-1} 2 r - r^{2} a_{0} &= 0, \quad a_{0} \neq 0 \\
r^{2} - r + r - V^{2} &= 0 \quad \Rightarrow \quad r = \pm V \\
\sum_{r+1} a_{(r+1)} r + a_{1} (r+1) - V^{2} a_{1} &= 0 \\
a_{1} (2r + 1) &= 0, \quad \text{we also know: } r = \pm V \\
\text{if } V = \pm \frac{1}{2} \rightarrow a_{1} \text{ is arbitrary.} \\
\text{if } V \neq \pm \frac{1}{2} \rightarrow a_{1} = 0 \\
\text{Assume } V \neq \pm \frac{1}{2} \text{ for now.}
\end{align*}
\]

\[
\begin{align*}
\sum_{n+r} \sum_{n \geq \frac{1}{2}} a_{n} \left[ (n+r)(n+r-1+1) - V^{2} \right] &= -a_{n-2} \\
a_{n} &= \frac{-a_{n-2}}{(n+r)^{2} - V^{2}} \quad (\ast)
\end{align*}
\]
Find the recursion for $r = \pm v$:

$$r = v : \quad a_n = \frac{-a_{n-2}}{(n+v)^2 - v^2} = \frac{-a_{n-2}}{n^2 + v^2 + 2nv - v^2} = \frac{-a_{n-2}}{n(n+2v)} \quad (n \geq 2)$$

$$a_2 = \frac{-a_0}{2(2+2v)} = \frac{-a_0}{2^2(1+v)}$$

$$a_4 = \frac{-a_2}{4(4+2v)} = \frac{-a_0}{3(2+2v) \cdot 2^2(1+v)}$$

$$a_6 = \frac{-a_4}{6(6+2v)} = \frac{-a_0}{3 \cdot 2(3+2v) \cdot 2^5(2+v)(1+v)}$$

The general expression:

$$a_{2m} = \frac{(-1)^m a_0}{m! \cdot 2^m (1+v)(2+v) \cdots (m+v)}$$
So, the first solution is:

\[ Y_1(x) = x^V \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m! \cdot 2^m (1 + V)(2 + V) \cdots (m + V)} \]  \hspace{1cm} (1)

Note: \( Y_1 \to 0 \) as \( x \to 0 \)

\[ r = -V: \quad a_n = \frac{-a_{n-2}}{(n-V)^2 - V^2} = \frac{-a_{n-2}}{n^2 + V^2 - 2nV - V^2} = \frac{-a_{n-2}}{n(n-2V)} \]

\( n > 2 \)

\( n = 2: \quad a_2 = \frac{-a_0}{2(2 - 2V)} = \frac{-a_0}{2 \cdot 2(1-V)} \)

\( n = 4: \quad a_4 = \frac{-a_2}{4(4 - 2V)} = \frac{a_0}{3 \cdot (2-V) \cdot 2^2 (1-V)} = \frac{a_0}{2 \cdot 2^4 (2-V)(1-V) \cdot 2^{2(V)}} \)

\( n = 6: \quad a_6 = \frac{-a_4}{6(6 - 2V)} = \frac{-a_4}{3 \cdot 2^2 (3-V) \cdot 6(6 - 2V)} = \frac{-a_0}{3 \cdot 2 \cdot 2^6 (1-V)(2-V)(3-V)} \)
or in general

\[ a_{2m} = \frac{(-1)^m a_0}{m! \frac{2^m}{2} (m-1)(2-1)(m-2)(m-3) \ldots (m-D)} \]

\[ y_2(x) = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m! \frac{2^m}{2} (m-1)(2-1)(m-2)(m-3) \ldots (m-D)} \]

Note: \( y_2 \to \infty \) as \( x \to 0 \)

**Bessel's function of order \( \nu = 0 \):**

If \( \nu = 0 \), then the indicial equation has a double root: \( r_{1,2} = 0 \).

The recursion is (from (\#)): \( \frac{\nu \nu}{\nu (\nu-1)} \) :

\[ a_n = \frac{-a_{n-2}}{(n+r)^2}, \quad n \geq 2, \quad a_1 = 0 \]

\[ a_2 = \frac{-a_0}{2}, \quad a_4 = \frac{-a_2}{4^2} = \frac{a_0}{2 \cdot 4^2} = \frac{a_0}{2^4 \cdot 2^2} \]

\[ a_6 = \frac{-a_4}{6^2} = \frac{-a_0}{2^4 \cdot 6^2} = \frac{-a_0}{2 \cdot (3!)^2} \]
\[ a_{2m} = \frac{(-1)^m a_0}{2m \binom{2}{m} (m!)} \]

So:

\[ y_1(x) = x^0 \sum_{m=0}^{\infty} \frac{(-1)^m a_0 x}{2m \binom{2m}{m} 2^2 (m!)} = J_0(x) \]

This is known as the Bessel function of the first kind of order zero.

Note that you could get \( J_0(x) \) by plugging \( \nu = 0 \) into \( y_1(x) \) or \( y_2(x) \) solutions in (1) and (2).

Now, we need to find a second solution.

Keep \( r \) as a parameter:

\[ a_n = -\frac{a_{n-2}}{(n+r)^2} \quad a_2 = -\frac{a_0}{(2+r)^2} \]

\[ a_4 = -\frac{a_2}{(4+r)^2} = \frac{a_0}{(2+r)^2 (4+r)^2} \]
So:

\[ y(x, r) = x^r \left[ 1 - \frac{x^2}{(2+r)^2} + \frac{x^4}{(2+r)^2(4+r)^2} + \cdots \right] \]

Find the second solution from:

\[ \frac{dy(x, r)}{dr} \bigg|_{r=0} = \ln(x) \cdot J_0(x) + x \left[ \frac{4x^2}{(2+r)^3} + \cdots \right] \bigg|_{r=0} \]

\[ \uparrow \]

remember: \( x = e \)

\[ = \ln|x| \cdot J_0(x) + \left( \frac{x^2}{4} + \cdots \right) \]

\[ = y_0(x) \]

\[ \uparrow \]

the Bessel function of the second kind of order zero.

(or a certain linear combination of \( J_0 \) and \( y_0 \))
The general solution is:

\[ y = C_1 J_0(x) + C_2 Y_0(x) \]

**Note:**

\( Y_0(x) \) has a logarithmic singularity at \( x = 0 \).

\[ x \to 0, \quad Y_0(x) \to -\infty \]

If the solution, \( y(x) \), is finite at \( x = 0 \), then \( C_2 = 0 \), i.e., \( Y_0(x) \) should be discarded.
Now, let’s see what happens when \( v = \pm \frac{1}{2} \): 

Note: the roots are: \( r = \pm \frac{1}{2} \), they are 1 (an integer apart).

\[
a_n = -\frac{a_{n-2}}{(n+r)^2 - 1/4} \quad \text{for } n > 2, \text{ and } a_1 \text{ is arbitrary}
\]

\[
r = \frac{1}{2} : \quad a_n = -\frac{a_{n-2}}{n + \frac{1}{4} + n - \frac{1}{4}} = -\frac{a_{n-2}}{n(n+1)}
\]

\[
r > 2, a_1 \neq 0
\]

\[
a_2 = -\frac{a_0}{3.2} = -\frac{a_0}{3!}
\]

\[
a_4 = -\frac{a_2}{5.4} = \frac{a_0}{5.4.3.2} = \frac{a_0}{5!}
\]

So, the first solution is:

\[
y_1(x) = x^{1/2} \left[ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots \right] = x^{1/2} \sin x
\]

\[
r = -\frac{1}{2} : \quad a_n = -\frac{a_{n-2}}{n^2 + \frac{1}{4} - n - \frac{1}{4}} = -\frac{a_{n-2}}{n(n-1)}
\]

\[
l > 2, a_1 \neq 0
\]
\[ a_2 = \frac{-a_0}{2.1}, \quad a_3 = \frac{-a_1}{3.2} \]

\[ a_4 = \frac{-a_2}{4.3} = \frac{a_0}{4.3.2.1} = \frac{a_0}{4!} \]

\[ a_5 = \frac{-a_3}{5.4} = \frac{a_1}{5.4.3.2} = \frac{a_1}{5!} \]

So, the second solution is:

\[ a_0 x^{-1/2} \left[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \right] + a_1 x^{-1/2} \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \right] \]

\[ = a_0 x^{-1/2} \cos x + a_1 x^{1/2} \left[ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots \right] \]

\[ = a_0 x^{-1/2} \cos x + a_1 x^{1/2} \sin x \]

\[ \text{this is the same as } y_1(x) \]

So: \[ y_2(x) = x^{1/2} \cos x \]

The general solution is:

\[ y = C_1 x^{-1/2} \cos x + C_2 x^{1/2} \sin x \]
If the roots of the indicial equation are separated by an integer, i.e.: \( r_1 = r_2 + N \), \( N \) is a positive integer, \( N \in \{1, 2, \ldots \} \), then the series:

\[
y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n
\]

are not necessarily linearly independent.

This happens when in Bessel's equation

\( \nu \in \{..., -1, 0, 1, 2, \ldots \} \).

In such cases, the second solution is found from:

\[
y_2(x) = ay_1(x) \ln|x| + \sum_{n=0}^{\infty} b_n x^{n+r_2}
\]

\((r_1 = r_2 + N > r_2)\)
Summary of Frobenius series solution:

Assume $x=0$ is a singular point of ODE:

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

If $Q(x) x$ and $R(x) x^2$ are analytic and have valid Taylor expansions, then $x=0$ is a regular singular point.

$$Q(x) x = P(x) = P_0 + P_1 (x-x_0) + \cdots$$

$$R(x) x^2 = Q(x) = q_0 + q_1 (x-x_0) + \cdots$$

where: $P_0 = \lim_{x \to 0} \frac{Q(x)}{P(x)} x$, $q_0 = \lim_{x \to 0} \frac{R(x)}{P(x)} x^2$

Then the indicial equation is:

$$r(r-1) + P_0 r + q_0 = 0$$
Assume the indicial equation has two real roots \( r_1, r_2 \), where \( r_1 > r_2 \).

Then the first solution always is:

\[
y_1(x) = |x|^{r_1} \left[ 1 + \sum_{n=1}^{\infty} a_n x^n \right]
\]

\[
= \sum_{n=0}^{\infty} a_n x^{n+r_1} \quad \text{if} \quad x > 0
\]

- \( a_n \) is found from a recursion by substitution into the ODE.

- \( a_0 \) is arbitrary.

Finding the second solution \( y_2(x) \):

(i) If \( r_1 - r_2 \neq 0 \) and \( r_1 - r_2 \neq N \) (\( N \) an integer)

Then the second solution has the form:

\[
y_2(x) = |x|^{r_2} \left[ 1 + \sum_{n=1}^{\infty} a_n x^n \right]
\]

\[
= \sum_{n=0}^{\infty} b_n x^{n+r_2} \quad \text{if} \quad x > 0
\]
(ii) If \( r_1 = r_2 \):

\[
y_2(x) = y_1(x) \ln|x| + |x|^r_1 \sum_{n=1}^{\infty} c_n x^n
\]

\[
= y_1(x) \ln x + \sum_{n=1}^{\infty} c_n x^{n+r_1} \quad \text{if } x > 0
\]

(iii) If \( r_1 = r_2 + N \), \( N \) a positive integer

then:

\[
y_2(x) = a y_1(x) \ln|x| + |x|^{r_2} \left[ 1 + \sum_{n=1}^{\infty} d_n x^n \right]
\]

\[
= a y_1(x) \ln x + \sum_{n=0}^{\infty} d_n x^{n+r_2} \quad \text{if } x > 0
\]

↑

note: there is a constant here

Note:

1. If \( r_1 \) and \( r_2 \) are complex, the form \( y_2 \)
in (ii) and \( y_1 \) are still valid. We need to convert complex-valued to real-valued solution, which needs a lot of algebra...
2. A summary of these solutions is given in the formula sheet for the exam.

3. The general solution is:

\[ y(x) = C_1 y_1(x) + C_2 y_2(x) \]
A list of variable coefficient ODES that occur in applied problems (from Brannan and Boyce).

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