Introduction to PDEs:

\( u \): dependent variable

\( x, y, z, t \): independent variables

Order of the PDE: the order of the highest partial derivative in the equation.

Example: \( u(x, y) \), \( u \) is an unknown function dependent on \( x, y \).

\[ a(x, y) u_x + b(x, y) u_y + c(x, y) u = d(x, y) \] (1)

is a linear first order ODE.

\[ A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \] (2)

A linear 2nd order ODE

\( A, B, C, \ldots, G \): constants or functions of \( x, y \)

If \( G = 0 \) the PDE is homogeneous, if \( G \neq 0 \) the PDE is non-homogeneous.
Consider the analogy with quadric surfaces:

\[ AX^2 + BXY + CY^2 + DX + EY = K \]

\[ \Delta = B^2 - 4AC \]

<table>
<thead>
<tr>
<th>( \Delta )</th>
<th>Type</th>
<th>Quadric</th>
<th>PDE</th>
<th>PDE Nature</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta = 0 )</td>
<td>Parabolic</td>
<td>( X^2 = T )</td>
<td>( u_t = u_{xx} )</td>
<td>Heat eqn. (diffusion eqn.)</td>
</tr>
<tr>
<td>( \Delta &lt; 0 )</td>
<td>Elliptic</td>
<td>( AX + CY = 1 )</td>
<td>( u_{xx} + u_{yy} = 0 )</td>
<td>Laplace's eqn.</td>
</tr>
<tr>
<td>( \Delta &gt; 0 )</td>
<td>Hyperbolic</td>
<td>( T - \alpha XY = 1 )</td>
<td>( u_{tt} = \alpha u_{xx} )</td>
<td>Poisson's eqn.</td>
</tr>
</tbody>
</table>

All linear 2nd order PDEs can be transformed into one of these types.
Example: Traffic flow (1D Conservative law)

Assume we are looking at the traffic flow at a length $\Delta x$ of a highway.

$u(x, t) = \text{density of cars at point } x \text{ at time } t$.

$[u] = \# \text{ of cars / unit length}$

$q(x, t) = \text{flux of cars at point } x \text{ at time } t$.

$[q] = \# \text{ of cars / unit time}$

\[ \begin{array}{c|c|c}
\text{flux in} & q(x, t) & q(x + \Delta t, t) \\
\hline
x & \Delta x & x + \Delta x
\end{array} \]

\[ u(x, t) \quad \quad \quad \quad \quad u(x + \Delta x, t) \]

\[ \text{flux out} \]
The conservation law tells us:

change in the # of cars over \([t, t+\Delta t]\)

\[ \# \text{ cars in} - \# \text{ cars out} \]

Or:

\[ u(x, t+\Delta t) \Delta x - u(x, t) \Delta x \]

\[ q(x, t) \Delta t - q(x+\Delta x, t) \Delta t \quad (\ast) \]

Check dimensions:

\[ \frac{\# \text{ cars}}{\Delta x} \cdot \frac{1}{t} = \frac{\# \text{ cars}}{\Delta x} \cdot \frac{1}{\Delta x} \]

Divide \((\ast)\) by \(\Delta t \cdot \Delta x\):

\[ \frac{u(x, t+\Delta t) - u(x, t)}{\Delta t} = \frac{q(x, t) - q(x+\Delta x, t)}{\Delta x} \]

Now let \(\Delta t \to 0\), \(\Delta x \to 0\):

\[ \frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x} \]

or:

\[ \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \]
This is conservation law PDE. 

\( u(x,t) \) and \( q(x,t) \) are both unknowns.

To be able to solve this PDE we need to know how \( q \) is related to \( u \).

This information comes from the nature of the problem. For instance it can be:
- an equation of state
- a constitutive law
- ... 

Let's assume:

\[ q = cu \quad , \quad c > 0 \] a linear flux-density relationship

i.e. when the number of cars increases, the flux increases proportionally.
So, we get:

\[ \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \]

or, use another notation:

\[ u_t + c u_x = 0 \]

Since the PDE has constant coefficients and is a linear combination of time and spatial partial derivatives, try a solution:

\[ u(x, t) = e^{ikx+6t} \]

\[ u_t = 6e^{ikx+6t} \]

\[ u_x = (ik) e^{ikx+6t} \]

\[ \text{sin/cos behaviour in } x \]

\[ \text{exponential damping in time } (\delta < 0) \]
Substitute into PDE:
\[ i k x + \sigma t \]
\[ (\sigma + i k c) e = 0 \]
or: \[ \sigma = -i k c \] "a dispersion relation"
So, the solution is:
\[ u(x,t) = e^{i k x - i k c t} \]
\[ u(x,t) = e \]

You can show that any differentiable function \( f \) with the functional form \( f(x - ct) \) is a solution to this PDE:
\[ u(x,t) = f(x - ct) \]
\[ u_t = -c f'(x - ct), \quad u_x = f'(x - ct) \]
\[ u_t + cu_x = -c f' + c f' = 0 \]
The PDE:

\[ u_t + cu_x = 0 \]

with the solution

\[ u(x,t) = f(x-ct) \]

can be interpreted as a right moving wave using the Galilean transformation:

The wave propagates in time to the right observer ①, stationary, sees: \( x \)

observer ②, moving with the wave, sees:

\[ x' = x - ct \]
\( u_t + cu_x = 0 \), Solution: \( u(x,t) = f(x-ct) \)
right moving wave

\( u_t - cu_x = 0 \), Solution: \( u(x,t) = f(x+ct) \)
left moving wave

Second order wave equation:
A wave that moves in both directions.

\[
\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u(x,t) = 0
\]

\[
\begin{align*}
\text{right moving} & \quad \text{left moving} \\
\text{wave operator} & \quad \text{wave operator}
\end{align*}
\]

\[
\left( \frac{\partial^2}{\partial t^2} - c \frac{\partial^2}{\partial t \partial x} + c \frac{\partial^2}{\partial x \partial t} - c^2 \frac{\partial^2}{\partial x^2} \right) u(x,t) = 0
\]

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0
\]

The second order wave equation.
Now, what happens if drivers slow down if they see an increase in car density ahead of them?

\[ q = cu - D u_x \]

\[ \uparrow \quad \leftarrow \quad \text{a constant} \]

\[ \text{increase in car density} \]

Check dimensions:

\[ \frac{\# \text{ cars}}{T} = \frac{M}{T} \cdot \frac{\# \text{ cars}}{M} = [D] \cdot \frac{\# \text{ cars}}{M^2} \]

So, \( D \) should have dimensions: \([D] = \frac{M^2}{T}\)

Substitute in the conservation PDE:

\[ \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \]

\[ q = cu - D u_x \]

\[ u_t + cu_{xx} = D u_{xx} \]

\[ \text{convection} \quad \text{diffusion} \quad \text{D: a diffusion coefficient} \]
So, now in addition to moving at speed $c$ to the right, the wave diffuses too.

Now, if you make a change of variable:

$z = x - ct$, i.e., you move with the center of the wave, and find the PDE for $U(z)$:

$$U_t = DU_{zz}$$

This means the observer that travels with the wave only sees the diffusion.
Finding the dispersion relation for the convection-diffusion equation:

Consider a solution: \( u(x, t) = e^{i(kx + \omega t)} \)

Substitute in:

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2}
\]

\[
(\omega + ick) e^{i(kx + \omega t)} = (-k^2)D e^{i(kx + \omega t)}
\]

\[
\omega = -i k c - k^2 D
\]

\( \omega \rightarrow \omega_d \) due to convection and diffusion

\[
\omega_d = -ikct - k^2 D
\]

\[ u(x, t) = e^{ik(x-ct)} e^{-k^2Dt} \]

Decay in time due to diffusion \( (D > 0) \)

right moving wave