Introduction to PDEs (cont'd):

The heat/diffusion equation:

Consider the heat conduction in a length $\Delta x$ of a conducting bar:

$u(x,t)$: The temperature at location $x$, time $t$, and has units degrees Kelvin, $[u] = K$

$q(x,t)$: The heat flux, or the flux of heat energy per unit area, $[q] = \frac{J}{m^2 \cdot s}$

$C$: The specific heat capacity. The amount of energy needed to increase the temperature of
one kilogram of the material \( \frac{J}{kg \cdot K} \) by one degree Kelvin

\[ c = \frac{J}{kg \cdot K} \quad \text{(a material property)} \]

\( p \): Density of the material, \( [p] = \frac{kg}{m^3} \)

\( A \): The cross sectional area of the bar

\[ [A] = m^2 \]

Now, let's write down the conservation of energy:

The increase in the thermal energy of the bar with length \( \Delta x \) = thermal energy in - thermal energy out
C. \[ u(x, t+\Delta t) - u(x, t) \] \cdot \rho \cdot A \cdot \Delta x.

\[ = \left[ q(x, t) - q(x+\Delta x, t) \right] A \cdot \Delta t \] (*)

Check the dimensions in equation above:

\[ \frac{J}{kg \cdot m^3} \cdot k \cdot \frac{kg \cdot m^2}{m^3} = \frac{J}{m^2 \cdot s} \cdot m^2 \cdot s \]

\[ J = J \]

Divide (*) by \( A \Delta x \cdot \Delta t \): 

\[ PC \left[ \frac{u(x, t+\Delta t) - u(x, t)}{\Delta t} \right] = \frac{\left[ q(x, t) - q(x+\Delta x, t) \right]}{\Delta x} \]

Let \( \Delta t \to 0 \) and \( \Delta x \to 0 \):

\[ PC \frac{du}{dt} = -\frac{dq}{dx} \quad \text{or:} \quad PC \frac{du}{dt} + \frac{dq}{dx} = 0 \]

The energy conservation PDE.
Now, we need to find a constitutive relation between $u$ and $q$. There are two ways to state a link between temperature and heat flux:

1) Fourier's law: $q = -k \frac{du}{dx}$

Heat always flows from higher temperature to lower temperature regions.

$k$: a thermal conductivity

$[K] = \frac{J}{k \cdot \text{m} \cdot \text{s}} = \frac{W}{k \cdot \text{m}}$

Substitution in the energy conservation PDE gives:

$\rho c \frac{du}{dt} - k \frac{d^2u}{dx^2} = 0 \quad \rightarrow \quad \frac{du}{dt} = \frac{k}{\rho c} \frac{d^2u}{dx^2}$

It is common to call $\alpha = \frac{k}{\rho c}$ a diffusion coefficient:

$[\alpha] = \frac{J \cdot \text{m}^2}{k \cdot \text{s} \cdot \text{m} \cdot \text{kg} \cdot \text{K}} = \frac{m^2}{s}$
2) Fick's law: \( q = -\alpha^2 \frac{\partial (p\cdot c\cdot u)}{\partial x} \)

The heat flux is from regions of high concentration of energy to regions of low concentration of energy. Here, \( p\cdot c\cdot u \) is the concentration of thermal energy, and has units: \( [p\cdot c\cdot u] = \text{kJ} \cdot \text{J} / \text{m}^3 \cdot \text{kg} \cdot \text{K} = \text{J} / \text{m}^3 \)

Substitution in:

\( p\cdot c \frac{du}{dt} + \frac{dq}{dx} = 0 \)

gives:

\[ \frac{du}{dt} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \]

The diffusion eqn.

\( \alpha^2 \) : the diffusion coefficient.

It can similarly be shown that the 2D diffusion equation is:

\[ \frac{du}{dt} = \alpha^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]
Another nice way of arriving at the diffusion equation: random walk - see lecture 7 of Prof. Peirce's lectures.

The wave equation: The motion of an elastic bar.

Consider an elastic bar:

\[ u(x,t) \] \hspace{1cm} \[ u(x+\Delta x,t) \]
\[ \sigma(x,t) \] \hspace{1cm} \[ \sigma(x+\Delta x,t) \]

\( \rho \): density \( [\rho] = \text{kg/m}^3 \)

\( u(x,t) \): displacement from equilibrium, \( [u] = \text{m} \)

\( \sigma(x,t) \): the normal stress, \( [\sigma] = \frac{\text{N}}{\text{m}^2} \)

\( A \): the cross-sectional area \( [A] = \text{m}^2 \)
Newton's second law:

\[
\left[ \delta (x + \Delta x, t) - \delta (x, t) \right] \cdot A = \rho \cdot \Delta x \cdot A \cdot \frac{\partial^2 u}{\partial t^2}
\]

\[ \uparrow \]\[ \text{net force} \]
\[ \uparrow \]\[ \text{mass} \]
\[ \text{acceleration} \]

Divide by \( A \cdot \Delta x \):

\[
\frac{\delta (x + \Delta x, t) - \delta (x, t)}{\Delta x} = \rho \frac{\partial^2 u}{\partial t^2}
\]

Or, in the limit \( \Delta x \to 0 \):

\[
\frac{\partial \delta}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}
\]

balance of linear momentum.

Now, we need a constitutive law that gives a relation between \( \delta \) and \( u \) to solve the PDE.
Hooke's law:

A relationship between the stress and the strain:

\[ \sigma = \frac{d\varepsilon}{dx} \]

\[ \varepsilon = \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} \]

\[ \varepsilon = \frac{du}{dx} \]

Substitute in the momentum balance:

\[ \frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \]

\[ E \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} \]

or

\[ \frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} \]
This is the wave equation, where the wave speed is:  
\[ c = \sqrt{\frac{E}{\rho}} \]

\[ u_{tt} = c^2 u_{xx} \] a 2D wave eqn.

\[ \left( \frac{\partial}{\partial t} + c^2 \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u = 0 \]

\[ u(x,t) = f(x-ct) + g(x+ct) \] a general form of the solution.

Laplace's equation: Flow in porous media

Consider the steady-state 2D flow in porous media.
\( u \): \( x \) component of velocity, \([u] = \text{m/s}\)

\( v \): \( y \) component of velocity, \([v] = \text{m/s}\)

\( \rho \): density \([\rho] = \text{kg/m}^3\)

\( q \): mass flux \([q] = \text{kg/s}\)

The conservation of mass tells us that the sum of fluxes through all boundaries should be zero:

\[ \rho \left[ u(x + \Delta x, y) - u(x, y) \right] \Delta y \cdot l + \rho \left[ v(x, y + \Delta y) - v(x, y) \right] \Delta x \cdot l = 0 \]

(\( l \): unit length in 3rd direction)
Check the dimensions of each term:

\[
\frac{\text{kg}}{\text{m}^3} \cdot \frac{\text{m}}{\text{s}} \cdot \frac{\text{m}}{\text{m}} = \text{kg/s} \quad \text{mass flux unit}
\]

Divide (*) by \( \rho \Delta x \Delta y \Delta l \):

\[
\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} = 0
\]

Let \( \Delta t \to 0, \Delta y \to 0 \):

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{Continuity equation}
\]

Now, we need to find a constitutive relation between \( u \) and \( v \).

For flow in porous media, you use Darcy's law as a constitutive law:

\[
u = -K \frac{\partial h}{\partial x}, \quad v = -K \frac{\partial h}{\partial y}
\]
where:

\( k \): hydraulic conductivity, \([k] = \frac{m}{s}\)

\( h \): hydraulic head, \([h] = m\)

So, Darcy’s law states that the flow direction is from regions with higher hydraulic head to regions with lower hydraulic head.

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Note: Darcy’s law is also sometimes stated as:

\[
U = -\frac{k}{\mu} \frac{\partial p}{\partial x}, \quad \text{where}
\]

\( k \): permeability, \([k] = m^2\)

\( \mu \): viscosity of fluid \([\mu] = \text{Pa} \cdot \text{s}\)

\( p \): pore pressure \([p] = \text{Pa}\)

Substituting \( U = -k \frac{\partial h}{\partial x} \) and \( V = -k \frac{\partial h}{\partial y} \)
into continuity eqn. gives:

\[
\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0
\]

Laplace’s eqn.