Separation of variables:

Consider the heat (diffusion) equation:

\[ u_t = \alpha^2 u_{xx} \]
\[ \alpha^2 : \text{the diffusion coefficient} \]

a parabolic PDE.

Assume an exponential solution:

\[ u(x,t) = e^{kx+\delta t} \]

\[ u_t = \delta e^{kx+\delta t}, \quad u_x = k e^{kx+\delta t}, \quad u_{xx} = k e^{kx+\delta t} \]

Substitute into the diffusion equation:

\[ \delta e^{kx+\delta t} = \alpha^2 k e^{kx+\delta t} \]

\[ \delta = \alpha^2 k \]

So the solution will have the form:

\[ u(x,t) = e^{kx} \cdot e^{\alpha^2 k t} \]

as \( t \to \infty \), this term goes to \( \infty \).

So, this cannot be a physical solution.
Now, try complex exponentials:

\[ u(x,t) = e^{ikx + \sigma t} \]

\[ u_t = \sigma e^{ikx + \sigma t}, \quad u_x = ike^{ikx + \sigma t}, \quad u_{xx} = -k^2 e^{ikx + \sigma t} \]

Substitute into diffusion equation:

\[ \sigma e^{ikx + \sigma t} = -k^2 e^{ikx + \sigma t} \]

or: \[ \sigma = -k^2 \]

So, the solution is: \[ u(x,t) = e^{ikx} \cdot e^{-k^2 t} \]

\[ \text{decays in time} \]

This solution is acceptable.

\[ \lambda = 2\pi / k \]

\[ \sin kx \]

\[ \sin k(x+\lambda) = \sin kx \]

\[ k: \text{wavenumber} \]

\[ \sigma: \text{the rate of decay in time} \]

\[ \sigma = -k^2 \]

"the dispersion relation"
\( k \), the wavenumber is determined by matching the boundary conditions.

Let's consider the initial-boundary value problem:

\[ u_t = \alpha^2 u_{xx} \quad \text{the diffusion equation} \]

\( 0 < x < L, \ t > 0 \)

Initial condition: \( u(x, 0) = f(x) \), \( 0 < x < L \)

Different boundary conditions:

1) Dirichlet problem:

\[ u(0, t) = 0 = u(L, t) \]

2) Neumann problem:

\[ u_x(0, t) = 0 = u_x(L, t) \]

\[ \sin \left( \frac{n \pi x}{L} \right) \]

\( n = 1, 2, \ldots \)
3) Mixed boundary conditions:

\[ u_0 = 0 = u(L, t) \]

\[ u_x = 0 \]

Solution to the Dirichlet problem by separation of variables:

\[ u_t = \alpha^2 u_{xx} \quad 0 < x < L, \ t > 0 \]

\[ u(0, t) = 0 = u(L, t) \quad \text{Dirichlet BC} \]

\[ u(x, 0) = f(x) \quad \text{I. C.} \]

Guess a solution:

\[ u(x, t) = X(x) \cdot T(t) \]
\[ u_t = \bar{X}(x) \cdot \dot{T}(t) \]
\[ u_{xx} = \bar{X}''(x) \cdot T(t) \]

So:
\[ \bar{X}(x) \cdot \ddot{T}(t) = \alpha^2 \bar{X}''(x) \cdot T(t) \]

Divide by \( \alpha^2 \bar{X} T \):
\[ \frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{\bar{X}''(x)}{\bar{X}(x)} = \text{Constant} = \lambda \]

This will give 2 ODEs.

\[ T(t) : \quad \dot{T}(t) = \lambda^2 T(t) \]
\[ T(t) = C e^{\lambda^2 t} \]

\[ \bar{X}(x) : \quad \bar{X}''(x) = \lambda \bar{X}(x) \]
\[ \bar{X}(0) = 0 = \bar{X}(L) \]

An obvious solution is \( \bar{X}(0) = 0 \), this is a trivial solution. Can we find a non-trivial solution?
The answer to this question depends on values of \( \lambda \rightarrow \) an eigenvalue problem.

1) if \( \lambda > 0 \):

\[
\lambda = \mu^2
\]

\[
X'' - \mu^2 X = 0, \quad X = e^{rx}
\]

\[
r^2 - \mu^2 = 0 \rightarrow r = \pm \mu
\]

\[
\begin{array}{r}
\text{Cosh}(\mu x) = \frac{e^{\mu x} + e^{-\mu x}}{2} \\
\text{Sinh}(\mu x) = \frac{e^{\mu x} - e^{-\mu x}}{2}
\end{array}
\]

\[
\begin{array}{c}
\begin{align*}
\text{or: } & X_- = A \text{ Sinh}(\mu x) + B \text{ Cosh}(\mu x) \\
X_- &= A \text{ Sinh}(\mu x) + B \text{ Cosh}(\mu x)
\end{align*}
\end{array}
\]

\[
X_- = A \text{ Sinh}(\mu x) + B \text{ Cosh}(\mu x)
\]

\[
X_- (0) = 0 \rightarrow B = 0
\]

\[
X_- (L) = 0 = A \text{ Sinh}(\mu L) = 0 \rightarrow A = 0
\]

\[
\begin{array}{c}
\text{trivial solution}
\end{array}
\]

or \( \mu = 0 \)

\[X_0 = A x + B\]

2) if \( \lambda = 0 \):

\[
\lambda = 0 \rightarrow X'' = 0, \quad X' = A
\]

\[
X = A x + B
\]
\[ \overline{x}(0) = B = 0, \quad \overline{x}(L) = A L = 0 \rightarrow A = 0 \]

This option also gives \( \overline{x} = 0 \), trivial solution.

\[ \text{II) } \lambda < 0 : \quad \lambda = -\mu^2 \rightarrow \overline{x}'' + \mu^2 \overline{x} = 0 \]

\[ \overline{x} = e^{rx} \rightarrow r^2 + \mu^2 = 0 \]

\[ r = \pm i \mu \]

\[ \overline{x} = A \sin(\mu x) + B \cos(\mu x) \]

\[ \overline{x}(0) = 0 \rightarrow B = 0 \]

\[ \overline{x}(L) = 0 \rightarrow A \sin(\mu L) = 0 \]

\[ \text{if } A = 0 \rightarrow \text{we get the trivial solution again.} \]

So, \( A \neq 0 \), \( \sin(\mu L) = 0 \)

or: \( \mu L = n \pi, \quad n = 1, 2, \ldots \)

\( \mu_n = \frac{n \pi}{L} \), \( n = 1, 2, \ldots \) are eigenvalues

\[ \overline{x}_n(x) = \sin \mu_n x = \sin \left( \frac{n \pi}{L} x \right) \] are eigenfunctions.
Shape of eigenfunctions:

\[ n=1 : \quad \sin \left( \frac{n\pi x}{L} \right) \]
\[ n=2 : \quad \sin \left( \frac{2n\pi x}{L} \right) \]
\[ n=3 : \quad \sin \left( \frac{3n\pi x}{L} \right) \]

So, the solutions will have the form:

\[ U_n(x, t) = e^{-\alpha^2 \left( \frac{n\pi}{L} \right)^2 t} \sin \left( \frac{n\pi x}{L} \right) \]

\[ n = 1, 2, 3, \ldots \]

A linear combination of these gives the general solution:

\[ U(x, t) = \sum_{n=1}^{\infty} b_n e^{-\alpha^2 \left( \frac{n\pi}{L} \right)^2 t} \sin \left( \frac{n\pi x}{L} \right) \]

How can we find the coefficients \( b_n \)?

From the initial condition: \( U(x, 0) = f(x) \)
\[ u(x,0) = f(x) = \sum_{n=0}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right) \]

a Fourier Series.