The one-dimensional wave equation:

\[ u_{tt} = c^2 u_{xx} \]

To solve this PDE we need **two initial conditions** (second order derivative in time) and **two boundary conditions** (second order time derivative in space), e.g.:

\[
\begin{align*}
  u(x,0) &= f(x) \quad \{ \text{I.C.'s} \} \\
  u_t(x,0) &= g(x) \\
  u(0,t) &= 0 \quad \{ \text{B.C.'s} \} \\
  u(L,t) &= 0
\end{align*}
\]

**Physical models (review):**

**Elastic bar:**

\[
\begin{align*}
  \frac{\partial b}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2} \\
  \frac{\partial b}{\partial x} &= E \frac{\partial^2 u}{\partial x^2}
\end{align*}
\]

balance of linear momentum

Constitutive law

\[ E : \text{Young's modulus} \]

\[ \frac{\partial u}{\partial x} = \varepsilon \text{ strain} \]
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad c = \sqrt{\frac{E}{\rho}} \]

Check units:

\[ [c^2] = \left[ \frac{\left( \frac{E}{\rho} \right)}{\rho} \right] = \frac{kg \cdot \frac{m}{s^2} \cdot \frac{1}{m^2}}{kg/m^3} = m^2/s^2 \]

Vibrating string:

No motion in the horizontal direction:

\[ T_1 \cos \theta_1 = T_2 \cos \theta_2 = T \]

Write Newton's second law for the vertical motion:

\[ T_2 \sin \theta_2 - T_1 \sin \theta_1 = \rho \Delta x \frac{\partial^2 u}{\partial t^2} \quad [\rho] = \frac{kg}{m} \]

Divide both sides by \( T \):

\[ \frac{\sin \theta_2}{\cos \theta_2} - \frac{\sin \theta_1}{\cos \theta_1} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} \]
\[\tan(\theta_1) = \left( \frac{\partial u}{\partial x} \right)_x, \quad \tan(\theta_2) = \left( \frac{\partial u}{\partial x} \right)_{x+\Delta x}\]

\[\Rightarrow \quad \frac{1}{\Delta x} \left( \frac{\partial u}{\partial x} \bigg|_{x+\Delta x} - \frac{\partial u}{\partial x} \bigg|_x \right) = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}\]

Let \(\Delta x \to 0\):

\[\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}\]

or:

\[\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c = \sqrt{\frac{T}{\rho}}\]

Check the units:

\[\left[ \frac{T}{\rho} \right] = \frac{\text{kg.m/s}^2}{\text{kg/m}} = \text{m/s}^2\]

Shallow water waves:

\[c = \sqrt{gh}\]

\[\left[ gh \right] = \text{m/s}^2 \cdot \text{m} = \text{m/s}^2\]

Example: Tsunami wave: ocean depth \(\approx 10\text{ m}\)

\[g \approx 10\text{ m/s}^2\]
Tsunami wave speed:

\[ c = \sqrt{10.10^3} \, \text{m/s} = 360 \, \text{km/hr} \]

The 1D wave equation can be decomposed into a right moving and a left moving wave:

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = 0 \]

\[ c \to \rightarrow \rightarrow c \]

Remember that we can look for an exponential solution:

\[ u(x,t) = e^{ikx + 6t} \]

\[ u_{tt} = c^2 u_{xx} \]

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\[ \Rightarrow 6^2 e^{ikx + 6t} = -k^2 c^2 e^{ikx + 6t} \]

\[ 6^2 = -k^2 c^2 \quad \text{or} \quad 6 = \pm ik \quad \text{the dispersion relation} \]

So, the guess solution has the form:

\[ u(x,t) = e^{ik(x+ct)} \]

This hints that \( x+ct \) has a physical significance.
Calilean transformation:

By using the Calilean transformation: \( x' = x \pm ct \), corresponding to a frame of reference moving to the right at speed \( c \), you find a solution to the wave equation:

\[
U(x,t) = g(x \pm ct)
\]

\[
U_{xx} = c^2 g'' , \quad U_{tt} = (\pm c)^2 g''
\]

\[
\Rightarrow U_{tt} - c^2 U_{xx} = c^2 g'' - c^2 g'' = 0
\]

D'Alembert's solution to the wave equation:

Consider the 1D wave equation:

\[
U_{tt} = c^2 U_{xx}
\]

Subject to initial conditions:

\[
U(x,0) = f(x) , \quad U_t(x,0) = g(x)
\]

Let the solution be the sum of a right moving and a left moving wave:

\[
U(x,t) = F(x-ct) + G(x+ct) = \rightarrow \rightarrow + \leftarrow \leftarrow
\]
Apply the I.C.'s:

\[ p(x) = u(x, 0) = F(x) + G(x) \]
\[ g(x) = u_t(x, 0) = -c F'(x) + c G'(x) \]

Integrate

\[ -c F(x) + c G(x) = \int_0^x g(s) \, ds + A \]  

(1)

\[ c F(x) + c G(x) = c \cdot F(x) \]  

(2)

Solve for \( F(x) \) and \( G(x) \):

(1) + (2) \Rightarrow \quad G(x) = \frac{1}{2} F(x) + \frac{1}{2c} \int_0^x g(s) \, ds + \frac{A}{2c}

(2) - (1) \Rightarrow \quad F(x) = \frac{1}{2} F(x) - \frac{1}{2c} \int_0^x g(s) \, ds - \frac{A}{2c}

So, the time-dependent solution is

\[ u(x, t) = F(x - ct) + G(x + ct) \]

\[ = \frac{1}{2} \left[ F(x - ct) + F(x + ct) \right] + \frac{1}{2c} \left[ \int_0^{x - ct} g(s) \, ds + \int_0^{x + ct} g(s) \, ds \right] - \frac{A}{2c} + \frac{A}{2c} \]

or:

\[ u(x, t) = \frac{1}{2} \left[ F(x - ct) + F(x + ct) \right] + \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) \, ds \]

D'Almbert's solution to wave eqn. on \((-\infty, \infty)\)
Space-time interpretations of D’Alambert’s solution:

\[ x = x_0 + ct \]

or: \[ x_0 = x - ct \]

the right propagating characteristic line

So, \( x_0 = x - ct \) and \( x_0 = x + ct \) are the characteristic lines that originate from \( x_0 \) at \( t=0 \).

right moving characteristics:

\[ x - ct = x_0 \rightarrow t = \frac{x}{c} - \frac{x_0}{c} \]

left moving characteristics:

\[ x + ct = x_0 \rightarrow t = -\frac{x}{c} + \frac{x_0}{c} \]

These are the lines along which initial information is propagated.
On the $x-t$ plane these

$$x + ct = x_0$$

Region of influence:

The information from $(x_1, 0)$ reaches this region. The region of influence is bounded by the lines $x + ct = x_1$ and $x - ct = x_1$. 
Domain of dependence:

The solution at \((x_0, t_0)\) depends on all values in the domain of dependence. This domain is bounded \(x = x_0 - ct_0\) and \(x = x_0 + ct_0\).

A simple example:

\[
\begin{align*}
\mathcal{U}(x, t) &= \mathcal{F}(x) & \text{initial displacement} \\
\mathcal{U}_t(x, t) &= 0 & \text{no initial velocity}
\end{align*}
\]

D'Alembert solution:

\[
\mathcal{U}(x, t) = \frac{1}{2} \left[ \mathcal{F}(x - ct) + \mathcal{F}(x + ct) \right]
\]

The displacement at this point is the average of the displacement at two points at \(t = c\).