Laplace's equation on rectangular domains with Neumann B.C.'s and semi-infinite domains.

Example: Neumann problem on a rectangle

Consider the domain with Flux boundary conditions:

\[ \Delta u = u_{xx} + u_{yy} = 0 \]

BC: \( u_x(0,y) = 0 = u_x(a,y) \)

\( u_y(x,0) = 0, \quad u_y(x,b) = f(x) \)

Opposite homogeneous B.C.'s are in \( x \) direction → an eigenvalue problem in \( x \)

Separation of variables:

\[ u(x,y) = X(x) \cdot Y(y) \]

\( X''Y + Y''X = 0 \rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda = -\mu^2 \)

\( X'' + \mu^2 X = 0 \) \{ \rightarrow \mu_n \in \{ 0, \frac{n\pi}{a} \}, \quad n=1,2,..... \}

\( X'(0) = 0 = X'(a) \) \{ \rightarrow X_n \in \{ 1, \cos(\frac{n\pi x}{a}) \} \}
Note: $\lambda > 0$ will give a trivial solution $X = 0$.

$Y'' - \mu^2 Y = 0$

$\mu \neq 0$

$Y = Acosh(\mu y) + Bsinh(\mu y)$

$Y' = A\mu sinh(\mu y) + B\mu cosh(\mu y)$

$Y'(b) = 0 \Rightarrow A\mu sinh(\mu b) + B\mu cosh(\mu b) = 0$

$\Rightarrow B = -A \frac{\sinh(\mu b)}{\cosh(\mu b)} = \tanh(\mu b)$

$Y(y) = A\cosh(\mu y) - A \frac{\sinh(\mu b)}{\cosh(\mu b)} \cdot \sinh(\mu y)$

$= A \left\{ \frac{\cosh(\mu y) \cosh(\mu b) - \sinh(\mu b) \sinh(\mu y)^2}{\cosh(\mu b)} \right\}$

$= \frac{A \cosh(\mu(b-y)) - B \sinh(\mu(b-y))}{\cosh(\mu b)}$

$\mu = 0$

$Y'' = 0$, $Y' = C$, $Y = Cy + D$

$Y'(b) = C = 0$, $Y_0 = D_0 \cdot 1$, $\mu_0 = 0$
So, the general solution is:

\[ u(x, y) = D_0 + \sum_{n=1}^{\infty} D_n \cosh(M_n(y-b)) \cos(M_n x) \]

where \( M_n = \frac{n\pi b}{a} \)

We need to apply the inhomogeneous B.C.:
\[ u_y(x, 0) = f(x) \]

to find coefficients \( D_n \):

Take partial \( y \) derivative of \( u(x, y) \):

\[ u_y(x, y) = 0 + \sum_{n=1}^{\infty} D_n M_n \sinh(M_n(y-b)) \cos(M_n x) \]

\[ u_y(x, 0) = f(x) = 0 + \sum_{n=1}^{\infty} -D_n M_n \sinh(M_n b) \cos(M_n x) \]

This is similar to the Fourier cosine expansion of \( f(x) \):

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(M_n x) \]

This problem has a solution only if:

\[ a_0 = \frac{2}{a} \int_0^a f(x) \, dx = 0 \]
From a physical point of view, a steady-state solution of the 2D heat conduction problem, i.e.

\[ u_t = 0 = u_{xx} + u_{yy} \]

is only possible if the influx of heat has a zero average:

\[ \int_0^a f(x) \, dx = 0. \]

If this is true, then \( D_n \) coeffs are found from:

\[ -D_n \frac{\sinh(M_n b)}{M_n} = a_n^f = \frac{2}{a} \int_0^a f(x) \cos(M_n x) \, dx \]

\[ \rightarrow D_n = \frac{a_n^f}{M_n \sinh(-M_n b)} \]

Substituting this into \( u(x,y) \) solution:

\[ u(x,t) = D_0 + \sum_{n=1}^{\infty} a_n \frac{\cosh(M_n (y-b))}{M_n \sinh(M_n(-b))} \cos(M_n x) \]

Still an arbitrary constant.

\[ M_n = \frac{nr \pi}{a} \quad n=1,2,1 \ldots \]
$u(x,y)$ is said to be known up to an arbitrary constant.

- If $u_0(x,y)$ is the steady-state solution of the heat conduction problem $u_t = u_{xx} + u_{yy}$, with initial condition: $u(x,y,0) = u_0(x,y)$:

  \[
  \frac{\partial}{\partial t} \iint_{0}^{b} u \, dx \, dy = 0
  \]

  \[
  \iint_{0}^{b} \iint_{0}^{a} u \, dx \, dy = \text{constant} = \iint_{0}^{b} \iint_{0}^{a} u_0(x,y) \, dx \, dy
  \]

  \[
  D_0 \cdot a \cdot b + \sum_{n=1}^{\infty} \frac{a_n}{\mu_n} \int_{0}^{b} \cosh(\mu_n(y-b)) \, dy = \frac{b}{\mu_n \sinh(-\mu_n b)} \int_{0}^{b} \cosh(\mu_n(y-b)) \, dy
  \]

  \[
  D_0 = \frac{1}{ab} \iint_{0}^{b} \iint_{0}^{a} u_0(x,y) \, dx \, dy
  \]
Example: Laplace's equation on a semi-infinite strip with mixed, inhomogeneous B.C.'s

\[ \Delta u = u_{xx} + u_{yy} = 0 \quad 0 < x < a \]
\[ 0 < y < \infty \]

\[ u(x,0) = \phi_o, \quad u(a,y) = \phi_i \]

\[ u(x,0) = f(x) \]

\[ u(x,y) \rightarrow \phi_o (x-a) + \phi_i \]

as \( y \rightarrow \infty \)

First, remove the inhomogeneous B.C.'s and transform the problem into a homogeneous B.C. problem:

Let \( w(x) = Ax + B \) satisfy the inhomog. B.C.:

\[ w(x) = Ax + B, \quad w_x = A \]

\[ w_x(0) = \phi_o \quad \rightarrow \quad A = \phi_o \]

\[ w(a) = \phi_i \quad \rightarrow \quad \phi_o a + B = \phi_i \quad \rightarrow \quad B = \phi_i - \phi_o a \]

\[ w(x) = \phi_o x + \phi_i - \phi_o a = \phi_o (x-a) + \phi_i \]
Now decompose the problem into \( w \) and \( v \):

Let \( u(x,t) = \omega(x) + v(x,t) \)

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \\
&= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{PDE}
\end{align*}
\]

\[
\begin{align*}
u(x,0) &= \phi(x) = \omega(x) + v(x,0) \\
\Rightarrow \quad v(x,0) &= \phi(x) - \omega(x)
\end{align*}
\]

\[
\begin{align*}
u(x,\infty) &= \phi_0 (x-a) + \phi_1 = \phi_0 (x-a) + \phi_1 + v(x,\infty) \\
\Rightarrow \quad v(x,\infty) &= 0
\end{align*}
\]

\[
\begin{align*}
u_x(0,y) &= \phi_0 = \omega_x(0) + v_x(0,y) \\
&= \phi_0 + v_x(0,y) \\
\Rightarrow \quad v_x(0,y) &= 0
\end{align*}
\]

\[
\begin{align*}
u(a,y) &= \phi_1 = \omega(a) + v(a,y) \\
&= \phi_1 + v(a,y) \\
\Rightarrow \quad v(a,y) &= 0
\end{align*}
\]
So, the B.V.P. for $V(x,t)$ is:

\[ V_{xx} + V_{yy} = 0 \quad 0 < x < a, \quad 0 < y < \infty \]

\[ V(x,0) = \Phi(x) - w(x) \]

\[ V(x,y) \to 0 \quad \text{as} \quad y \to \infty \]

\[ V_x(0,y) = 0 = V(a,y) \]

Separation of variables:

Let: \( V(x,y) = \tilde{X}(x) \tilde{Y}(y) \)

\[ V_{xx} + V_{yy} = \tilde{X}'' \tilde{Y} + \tilde{Y}'' \tilde{X} = 0 \]

\[ \frac{\tilde{X}''}{\tilde{X}} = -\frac{\tilde{Y}''}{\tilde{Y}} = \lambda = -\mu^2 \]

( an eigenvalue problem in \( x \) )

\[ X'' + \mu^2 X = 0 \]

\[ X'(0) = 0 = X(a) \]

\[ X_n = \cos(\mu_n x) \quad \mu_n = \frac{(2n-1)\pi}{2a}, \quad n = 1, 2, ... \]
\[ y'' - \mu^2 y = 0 \implies y = C \cosh(\mu y) + D \sinh(\mu y) \]

To apply the \( y \to \infty \) it is easier to write \( y \) as exponentials:

\[ y = Ae^{\mu y} + Be^{-\mu y} \]

\( y \to 0 \) as \( y \to \infty \): \( A = 0 \)

So, \( Y_n(y) = e^{-\mu_n y} \)

\( V_n(x,y) = e^{-\mu_n y} \cdot \cos(\mu_n x) \), \( \mu_n = \frac{(2n-1)\pi}{2a} \)

\( n = 1, 2, 3, \ldots \)

\[ V(x,y) = \sum_{n=1}^{\infty} B_n e^{-\mu_n y} \cos(\mu_n x) \]

Apply the inhomogeneous B.C. \( V(x,0) = f(x) - w(x) \) to find \( B_n \) co-effs:

\[ V(x,0) = f(x) - w(x) = \sum_{n=1}^{\infty} B_n \cos(\mu_n x) \]
So, $B_n$ are Fourier cosine coefficients of $f(x) - \omega(x)$:

$$B_n = \frac{2}{a} \int_0^a \left[ f(x) - \omega(x) \right] \cos \left( \frac{(2n-1)\pi x}{2a} \right) \, dx$$

The final solution is:

$$u(x, y) = \phi_0 (x-a) + \Phi_1 + \sum_{n=1}^{\infty} B_n e^{-\left( \frac{(2n-1)\pi y}{2a} \right)} \cos \left( \frac{(2n-1)\pi x}{2a} \right)$$