Math 257/316 Assignment 8, 2018
1D Wave equation practice problems - Not for submission

**Problem 1:** The motion of a string subject to a gravitational load satisfies the following initial-boundary value problem:

\[
\begin{align*}
    u_{tt} &= c^2 u_{xx} - g, \quad 0 < x < 1, \quad t > 0 \\
    u(0,t) &= u(1,t) = 0 \\
    u(x,0) &= \sin(\pi x), \quad u_t(x,0) = 0.
\end{align*}
\]  

Here \( g \) is the acceleration due to gravity, which you may assume is constant.

(a) Determine the static deflection of the string, which is determined by solving (1) in which it is assumed that \( u_{tt} = 0 \) subject to the boundary conditions (2).

(b) Use the solution obtained in (a) to reduce the initial-boundary value problem to solving a homogeneous wave equation subject to homogeneous boundary conditions. Now use separation of variables to determine the solution to this boundary value problem and hence the complete solution of the entire initial-boundary value problem.

**HINT:** The following integral may be useful:

\[
\int_0^1 (x^2 - x) \sin(n \pi x) \, dx = 2 \frac{\cos(n \pi) - 1}{n^3 \pi^3}
\]

**Problem 2:** The motion of a string on an elastic foundation with a stiffness \( \gamma \) satisfies the following initial-boundary value problem:

\[
\begin{align*}
    u_{tt} &= u_{xx} - \gamma u, \quad 0 < x < 1, \quad t > 0 \\
    u(0,t) &= u(1,t) = 0 \\
    u(x,0) &= 0, \quad u_t(x,0) = g(x)
\end{align*}
\]

(a) Solve (3) subject to the boundary conditions (4) using separation of variables.

(b) For \( \gamma = 7 \pi^2 \) and \( g(x) = \sin 3 \pi x \), sketch the solution at \( t = 3/8 \).

**Problem 3:** Solve the following inhomogeneous initial boundary value problem for the wave equation:

\[
\begin{align*}
    u_{tt} &= c^2 u_{xx} + e^{-t} \sin(3x) + 1, \quad 0 < x < \frac{\pi}{2}, \quad t > 0 \\
    u(0,t) &= \frac{t^2}{2} \text{ and } u_x(\frac{\pi}{2},t) = t, \quad t > 0 \\
    u(x,0) &= 0, \quad u_t(x,0) = \sin(5x) + x, \quad 0 < x < \frac{\pi}{2}
\end{align*}
\]
Problem 4: Solve the following inhomogeneous initial boundary value problem for the wave equation:

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + e^{-t} \sin(5x), \quad 0 < x < \frac{\pi}{2}, \quad t > 0 \]

\[ u(0, t) = 0 \text{ and } u_x(\frac{\pi}{2}, t) = t, \quad t > 0 \]

\[ u(x, 0) = 0, \quad u_t(x, 0) = \sin(3x) + x, \quad 0 < x < \frac{\pi}{2} \]