A summary of heat and wave equations with inhomogeneous B.C.'s and source terms:

1. No source term, inhomogeneous (time dependent or time independent) B.C.'s:

   heat: \( U_t = \alpha^2 U_{xx} \)  
   wave: \( U_{tt} = c^2 U_{xx} \)

   1(a): Dirichlet B.C.: \( U(0,t) = \phi_0(t) \)  
   \( U(L,t) = \phi_1(t) \)

   guess \( w(x,t) = A(t)x + B(t) \)

   1(b): Mixed B.C.'s: \( U_x(0,t) = q_0(t) \)  
   \( U(L,t) = \phi_1(t) \)

   guess \( w(x,t) = A(t)x + B(t) \)

   1(c): Neumann B.C.: \( U_x(0,t) = q_0(t) \) (time-dependent)  
   \( U_x(L,t) = q_1(t) \)

   guess: \( w(x,t) = A(t)x^2 + B(t)x \)

   valid 1(d): Neumann B.C.: \( U_x(0,t) = q_0 \) (constant)  
   \( U_x(L,t) = q_1 \)

   guess: \( w(x,t) = Ax^2 + Bx + Ct \)
Decompose the problem into:

\[ u(x,t) = w(x,t) + v(x,t) \]

Find the B.V.P. for \( v(x,t) \), with homogeneous B.C.'s, and solve it (the process of removing the time-dependent B.C./s may create a source/sink term, see Problem 2).

2. Time-dependent source/sink terms with inhomogeneous time-dependent B.C.'s

Step 1: remove the inhomogeneous B.C.'s by guessing a \( w(x,t) \) as in problem 1 and decomposing the problem as:

\[ u(x,t) = w(x,t) + v(x,t) \]

Step 2: Solve the \( v(x,t) \) problem with homogeneous B.C.'s and \( \phi(x,t) \) term using the eigenfunction expansion method.
3. Time-independent Source/Sink with inhomogeneous B.C.'s:

3(a) - Time-independent B.C.'s:

Guess a steady-state solution $\omega_t = 0$, that satisfies the PDE with source/sink and B.C.'s

3(b) - Time-dependent B.C.'s:

First remove time-dependent B.C.'s using methods in problem 1. This might create time-dependent source/sink terms $\Rightarrow$ same as problem 2.
4 - Heat equation with heat loss / Wave equation on an elastic foundation with stiffness:

\[ u_t = u_{xx} - \beta^2 u \]
\[ u_{tt} = u_{xx} - \delta u \]

(a) with time-independent B.C.'s or source/sink term

First, assume a steady-state solution \( u_t = 0 \) to remove the source/sink term & inhomogeneous B.C.'s. \( \rightarrow \) decompose: \( u = w + V \)

Second, use separation of variables to solve the \( V(x, t) \) problem.

(b) with homogeneous B.C.'s and no source/sink term.

Use separation of variables directly.

( Group the term with \( \beta^2 \) or \( \delta \) with the time-dependent problem. )
with time-dependent source/sink
or time-dependent B.C.'s:

- Remove the inhomogeneous
  B.C.'s (Problem 1). This
  may create more time-dependent
  source/sink terms.

- Solve the problem with
  homogeneous B.C.'s and
  no source/sink terms.
  (problems 4(a),(b))

- Finally, use the eigenfunction
  expansion method to solve
  for the problem with
  source/sink terms.