Problem 1: 1D wave equation (Submit) Consider an infinite string subject to the initial conditions
\[ u(x, 0) = f(x) = \begin{cases} 
  x + 1 & \text{if } -1 \leq x < 0 \\
  1 - x & \text{if } 0 \leq x \leq 1 \\
  0 & \text{if } |x| > 1 
\end{cases} \]
\[ u_t(x, 0) = g(x) = 0. \]
Assume that for this string \((T/\rho)^{1/2} = 1\), with \(T\) being the tension in the string and \(\rho\) the density of the string per its unit length.
(a) In the x-t plane draw the important characteristics that can be interpreted from the D’Alembert’s solution. Using these characteristics identify different regions in the x-t plane that have the same expression for the displacement of the string. Find the expression for the displacement in each region. Then, use this geometric interpretation of the D’Alembert’s solution to sketch the shape of the string at times \(t = 0, 1/2, 1\) and \(2\).
(b) Use the Matlab code attached to this assignment to solve this problem numerically. Attach the plots for the shape of the string at any time to your assignment. Note that the small-scale oscillations in these plots are numerical artifacts and not physical. Compare the location of the peaks of the waves at each time from the numerical solution and D’Alembert’s solution.

Problem 2: Laplace’s equation on a rectangular domain (Submit) Consider the static deflection of a plate that satisfies the Laplace’s equation and is subject to the following boundary conditions
\[ u_{xx} + u_{yy} = 0 \text{ for } 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2 \]
\[ u_x(0, y) = 0.1, \quad u_y(x, 0) = 0 \]
\[ u(x, 2) = f(x) = \begin{cases} 
  x & \text{if } 0 \leq x < 1 \\
  2 - x & \text{if } 1 \leq x < 3 
\end{cases} \]
\[ u(3, y) = f(y) = \begin{cases} 
  y & \text{if } 0 \leq y < 1 \\
  2 - y & \text{if } 1 \leq y < 2 
\end{cases} \]
(a) Use the method of separation of variables to solve this boundary value problem.
(b) Verify that the solution you found satisfies all the boundary conditions.
(c) Use the Matlab code attached to this assignment to solve this problem numerically.
Attach the plots of the solution to your assignment. Compare the value of the solution for \( u(0,0) \) from the Matlab code and your analytical solution.

**Problem 3: Laplace’s equation on circular domains (Submit)** A metal plate occupies a quarter-annular region \( 0 < a \leq r \leq b \) and \( 0 \leq \theta \leq \pi/2 \). The horizontal face is insulated while the vertical face is kept at 2 degrees and the outer hoop is maintained at 0 degrees. The inner hoop is maintained at a temperature of \( \cos(2\theta) \). Determine the steady state temperature by solving the following BVP in \( \Omega \):

\[
\begin{align*}
\frac{v_{rr}}{r} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta \theta} &= 0 & \text{in } \Omega \\
v(0,0) &= 0 & \text{for } a < r < b \\
v(r, \pi/2) &= 2 & \text{for } a < r < b \\
v(a, \theta) &= \cos 2\theta & \text{for } 0 < \theta < \pi/2 \\
v(b, \theta) &= 0 & \text{for } 0 < \theta < \pi/2
\end{align*}
\]

**Problem 4 (Do not submit):** Consider the BVP:

\[
\begin{align*}
\phi'' + 6\phi' + \lambda \phi &= 0, & 0 < x < L \\
\phi(0) &= 0 \\
\phi(L) &= 0
\end{align*}
\]

(a) Put this BVP into Sturm-Liouville form.
(b) Compute all eigenvalues and eigenfunctions.
(c) Show explicitly that the eigenfunctions are mutually orthogonal. (Don’t forget to include the weight function inside the integral.)

**Problem 5 (Do not submit):** Consider the eigenvalue problem

\[
\begin{align*}
x^2 y'' + xy' + \lambda y &= 0 \\
y(1) &= 0 = y'(2)
\end{align*}
\]

a. Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions.
b. Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace’s equation on the quarter-annular region:

\[
\begin{align*}
\frac{u_{rr}}{r} + \frac{u_r}{r} + \frac{1}{r^2} u_{\theta \theta} &= 0, & 1 < r < 2, \quad 0 < \theta < \pi/2 \\
u(r,0) &= 0 \quad \text{and} \quad \frac{\partial u(r, \pi/2)}{\partial \theta} = f(r) \\
u(1, \theta) &= 0 \quad \text{and} \quad \frac{\partial u(2, \theta)}{\partial r} = 0
\end{align*}
\]

**Problem 6 (Do not submit):**
Solve the following heat conduction problem:

\[
\begin{align*}
\frac{u_t}{x} &= x^2 u_{xx} + 4x u_x & \text{for } x \in (1,2), \quad t > 0 \\
u(1,t) &= 1 \\
u(2,t) &= 1 \\
u(x,0) &= 1 - 5x^{-3/2}
\end{align*}
\]