Examples in the assignments:

5b) \( \sinh(x) y'' + xy' + y = 0 \) \[\implies \begin{cases} P(x) = \sinh x \\ Q(x) = x \\ R(x) = 1 \end{cases} \]

\( x = 0 \) : a singular point

Classify the singular point:

\[ \lim_{x \to x_0} \frac{Q(x)}{P(x)} = \lim_{x \to 0} \frac{x}{\sinh(x)} \]

\[ = \lim_{x \to 0} \frac{x}{\frac{2}{\sinh(x)}} \]

L'hospital's rule:

\[ = \lim_{x \to 0} \frac{2x}{\cosh(x)} = 0 \text{ finite} \]

\[ \lim_{x \to x_0} \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 \frac{1}{\sinh(x)} \]

\( x_0 = 0 \) is a r.s.p.

5d) \( y'' + \sqrt{x} y' - y = 0 \) \[\implies \begin{cases} P(x) = 1 \\ Q(x) = \sqrt{x} \\ R(x) = 1 \end{cases} \]

\[ \frac{Q(x)}{P(x)} = \sqrt{x} \text{ not analytic at } x_0 = 0 \implies x_0 = 0 \text{ is a singular point.} \]
\[(x-x_0) \cdot \frac{Q(x)}{P(x)} = x \cdot \sqrt{x}\]

is not analytic at \(x_0 = 0\).

So, \(x_0 = 0\) is an irregular singular point.
Frobenius series near regular singular points:

Consider the ODE:

\[ P(x) y'' + Q(x) y' + R(x) y = 0 \]

If \( x_0 \) is a regular singular point of the ODE, the Frobenius solution near this point has the form:

\[ y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

For the solution to be complete, we need to:

1. Find the values of \( r \)

2. Find the recursion relation for \( n \)

3. Find the radius of convergence of \( \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} \)
Example:

\[ Ly = 2x^2 y'' - xy' + (1-x)y = 0 \quad (1) \]

\[ x_0 = 0 \] is a singular point.

\[ \lim_{x \to 0} - \frac{x}{2x^2} = \frac{-1}{2} = p_0 \]
\[ x_0 = 0 \] is a regular singular point.

\[ \lim_{x \to 0} \frac{(1-x)}{2x^2} = \frac{1}{2} = q_0 \]

First method: Solve the approximate equation:

(a sequence of equations)

As \( x \to 0 \) near the singular point

\[ 1-x \to 1 \quad \text{or} \quad y \gg xy \]

So you can approximately say:

\[ L_0 y = 2x^2 y'' - xy' + y = xy \quad \text{near} \]

which is a Cauchy-Euler eqn.

\[ y = x^r \rightarrow [2r(r-1) - r + 1] x^r = 0 \]

\[ 2r^2 - 3r + 1 = 0 \quad \rightarrow \quad r = \frac{1}{2}, 1 \]

\[ y_0 = C_1 x^{1/2} + C_2 x \]
We need to correct this solution to take into account the forcing on the RHS of (2):

\[ L_1 y = 2x^2 y'' - xy' + y = c_1 x^{3/2} + c_2 x^2 \quad (3) \]

guess a particular solution:

\[ y_p = A_0 x^{3/2} + B_0 x^2 \]
\[ y'_p = \frac{3}{2} A_0 x^{1/2} + 2 B_0 x \]
\[ y''_p = \frac{3}{4} A_0 x^{-1/2} + 2 B_0 \]

Substitute in (3):

\[ \frac{3}{2} A_0 x^{3/2} + 4 B_0 x^2 - \frac{3}{2} A_0 x^{3/2} - 2 B_0 x^2 \]
\[ + A_0 x^{3/2} + B_0 x^2 = c_1 x^{3/2} + c_2 x^2 \]

Collect all coefficients of \( x^{3/2} \) and \( x^2 \) and make them equal on both sides:

\[ \frac{3}{2} A_0 - \frac{3}{2} A_0 + A_0 = c_1 \quad \rightarrow \quad A_0 = c_1 \]
\[ 4 B_0 - 2 B_0 + B_0 = c_2 \quad \rightarrow \quad B_0 = \frac{c_2}{3} \]
So: \[ y_1(x) = C_1 x^{1/2} (1+x) + C \frac{x(1+x)}{3} \]

the first two terms of the power series.

You can substitute \( y_1(x) \) in (2) and get the higher order terms in power series.

Method 2: We can also use the Frobenius solution from the start:

\[ 2x^2 y'' - xy' + (1-x) y = 0, \quad p(x) = 2x^2 \]
\[ q(x) = -x \]
\[ R(x) = 1-x \]

\[ p_0 = \lim_{x \to 0} x \frac{q(x)}{p(x)} = -\frac{1}{2} \]

\[ q_0 = \lim_{x \to 0} x^2 \frac{R(x)}{p(x)} = \frac{1}{2} \]

The corresponding Cauchy-Euler (or equidimensional) equation is:

\[ x^2 y'' - \frac{1}{2} x y' + \frac{1}{2} y = 0 \]

which has solutions:

\[ x^1, x^{1/2} \]
The Frobenius solution is:

\[ y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \]

\[ y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \]

\[ y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \]

Substitute in the ODE:

\[ 2x^2 y'' - xy' + (1-x) y = 0 \]

\[ 2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} \]

\[ - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^n = 0 \]

Shift the index\[
\begin{align*}
n+r+1 &= m+r \\
n+1 &= m \\
n &= m-1 \\
n = 0 &\rightarrow m = 1
\end{align*}
\]

Also take the \( n=0 \) terms out of the first three series:
\[ 2a_0 r^{(r-1)} x^r - a_0 r x^r + a_0 x^r \]

\[ + \sum_{n=1}^{\infty} \left[ 2a_n (n+r)(n+r-1) - a_n (n+r) + a_n - a_{n-1} \right] x^n \]

\[ = 0 \]

\[ x^n, \quad n > 1 \]

\[ a_n \left( 2(n+r)(n+r-1) - (n+r)+1 \right) = a_{n-1} \]

\[ a_n = \frac{a_{n-1}}{(n+r)(2(n+r)-3) + 1} \]

The indicial equation

\[ 2r(r-1) - r + 1 = 0 \]

\[ \Rightarrow 2r^2 - 3r + 1 = 0 \]

\[ r = 1, \quad \frac{1}{2} \]

This is the same as the characteristic equation of the corresponding C-E eqn.

The recursion relation

We have to find the recursion for \( r_1 = 1 \), and \( r_2 = \frac{1}{2} \).
\[ r_1 = \frac{1}{2} : \quad a_n = \frac{a_{n-1}}{(n+\frac{1}{2})(2n+1-3)+1} \]

\[ = \frac{a_{n-1}}{2(n+\frac{1}{2})(n-1)+1} \]

\[ = \frac{a_{n-1}}{2(n^2-\frac{1}{2}n-\frac{1}{2})+1} \]

\[ = \frac{a_{n-1}}{2n^2-n-1+1} = \frac{a_{n-1}}{n(2n-1)} \]

\[ n=1 : \quad a_1 = \frac{a_0}{1.1} \]

\[ n=2 : \quad a_2 = \frac{a_1}{2.3} = \frac{a_0}{2.3} = \frac{a_0}{6} \]

\[ n=3 : \quad a_3 = \frac{a_2}{3.5} = \frac{a_0}{15.6} = \frac{a_0}{90} \]

\[ y_1(x) = a_0 \cdot x^{1/2} \left[ 1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \ldots \right] \]

\[ r_2 = 1 : \quad a_n = \frac{a_{n-1}}{(n+1)(2(n+1)-3)+1} \]

\[ = \frac{a_{n-1}}{(n+1)(2n-1)+1} \]

\[ = \frac{a_{n-1}}{2n^2+n-1+1} \]
\[ a_n = \frac{a_{n-1}}{n(2n+1)} \text{, for } n \geq 1 \]

\[ n = 1 : \quad a_1 = \frac{a_0}{1.3}, \quad n = 2 : \quad a_2 = \frac{a_1}{2.5} = \frac{a_0}{30} \]

\[ n = 3 : \quad a_3 = \frac{a_2}{3.7} = \frac{a_0}{630} \]

\[ y_2(x) = a_0 x^2 \left( 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \cdots \right) \]

**Note:** Compare this to the solution from previous method.

The last step is to find radius of convergence of series:

**Ratio test:**

\[
\lim_{n \to \infty} \left| \frac{a_n x^n}{a_{n-1} x^{n-1}} \right| = \lim_{n \to \infty} |x| \cdot \left| \frac{1}{(n+r)((2n+r)-3)+1} \right| = 0
\]

So, \( r = \infty \) for all \( x \).
The general solution is:

$$y(x) = c_1 x^{1/2} \left[ 1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \ldots \right]$$

$$+ c_2 x \left[ 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \ldots \right]$$

Examples of RSP, ISP and radius of convergence:

Example 1:

$$x(1+x^2)y'' + 2xy' + (x^2 + 1) y = 0$$

a- Find and classify the singular points.

b- What is the lower bound on the radius of convergence of power series expansion about $$x_0 = 2$$?

a- Singular points:

$$p(x) = x(1+x^2), \quad q(x) = 2x, \quad r(x) = x^2 + 1$$
\[ \frac{Q}{P} = \frac{2x}{x(1+x^2)} = \frac{2}{1+x^2} \rightarrow x_0 = \pm i \]

singular points

\[ \frac{R}{P} = \frac{x+1}{x(x^2+1)} = \frac{1}{x} \rightarrow x_0 = 0 \text{ as a singular point} \]

Classify the singular points:

\[ x_0 = 0 : \]

\[ \lim_{x \to 0} \frac{Q}{P} x = \lim_{x \to 0} \frac{2x}{1+x^2} = 0 \rightarrow x_0 = 0 \text{ is a regular singular point} \]

\[ \lim_{x \to 0} \frac{R}{P} x^2 = \lim_{x \to 0} \frac{x^2}{x} = 0 \]

\[ x_0 = i : \]

\[ \lim_{x \to i} \frac{Q}{P} (x-i) = \lim_{x \to i} \frac{2(x-i)}{1+x^2} = \frac{2}{2i} = \frac{1}{i} \text{ (hospital's rule)} \rightarrow \lim_{x \to i} \frac{2}{2x} = \frac{1}{i} \text{ finite} \]

\[ \lim_{x \to i} \frac{R}{P} (x-i)^2 = \lim_{x \to i} \frac{1}{x} (x-i)^2 = 0 \quad x_0 = i \text{ is a regular singular point} \]

\[ x_0 = -i : \]

\[ \lim_{x \to -i} \frac{Q}{P} (x+i) = \lim_{x \to -i} \frac{2(x+i)}{1+x^2} = \lim_{x \to -i} \frac{2}{2x} = -\frac{1}{i} \]

\[ \lim_{x \to -i} \frac{R}{P} (x+i)^2 = \lim_{x \to -i} \frac{(x+i)^2}{x} = 0 \quad x_0 = -i \text{ a RSP} \]
radius of convergence near $x_0 = 2$:
The distance from $x_0 = 2$ to the nearest singular point. You have to find this distance on the complex plane:

$$\sqrt{2^2 + 1} = \sqrt{5}$$

$p = 2$: the lower bound for the radius of convergence.
Example:

\[ \cos x y'' + y' + \cot(x) y = 0 \]

Find the singular points of this ODE. Classify them as regular or irregular.

\[ \frac{Q}{P} = \frac{1}{\cos x}, \quad \frac{R}{P} = \frac{\cos x}{\sin x \cos x} \cdot \frac{1}{\sin x} = \frac{1}{\sin x} \]

Singular points:

\[ \cos x = 0 \rightarrow x_o = (2n+1) \frac{R}{2} \]
\[ \sin x = 0 \rightarrow x_o = mR \]

Classification:

\[ \lim_{x \to (2n+1) \frac{R}{2}} \frac{(x - (2n+1) \frac{R}{2})}{\cos x} = \lim_{x \to (2n+1) \frac{R}{2}} \frac{1}{-\sin x} = (-1) < \infty \]

\[ \lim_{x \to (2n+1) \frac{R}{2}} \frac{(x - (2n+1) \frac{R}{2})^2}{\sin x} = 0 \]

So, \( x = (2n+1) \frac{R}{2} \) are regular singular points.
\[
\lim_{{x \to m\pi}} \frac{{(x - m\pi)}}{{\cos x}} = 0
\]

\[
\lim_{{x \to m\pi}} \frac{{(x - m\pi)^2}}{{\sin x}} = \lim_{{x \to m\pi}} \frac{{2(x - m\pi)}}{{\cos x}} = 0
\]

\[x = m\pi\] are also RSP.