Separation of variables:

Consider the heat (diffusion) equation:

$$u_t = \alpha^2 u_{xx} \quad \alpha^2 : \text{the diffusion coefficient}$$

a parabolic PDE.

Assume an exponential solution: $u(x,t) = e^{kx+bt}$

$$u_t = 6 e^{kx+bt} \quad , \quad u_x = ke^{kx+bt} \quad , \quad u_{xx} = k^2 e^{kx+bt}$$

Substitute into the diffusion equation:

$$6 e^{kx+bt} = \alpha^2 k e^{kx+bt}$$

$$\Rightarrow 6 = \alpha^2 k$$

So the solution will have the form:

$$u(x,t) = e^{kx+2k^2t}$$

as $t \to \infty$, this term goes to $\infty$.

So, this cannot be a physical solution.
Now, try complex exponentials:

\[ u(x, t) = e^{i k x + \sigma t} \]

\[ u_t = \sigma e^{i k x + \sigma t}, \quad u_x = i k e^{i k x + \sigma t}, \quad u_{xx} = -k^2 e^{i k x + \sigma t} \]

Substitute into diffusion equation:

\[ \sigma e^{i k x + \sigma t} = -k^2 e^{i k x + \sigma t} \]

or:

\[ \sigma = -k^2 \]

So, the solution is:

\[ u(x, t) = e^{-k^2 t} e^{i k x} \]

This solution is acceptable.

- \( \lambda = \frac{2\pi}{k} \)

- \( \sin kx \)

- \( \sin k(x+\lambda) = \sin kx \)

- \( k \): wavenumber

- \( \sigma \): the rate of decay in time

\[ \sigma = -k^2 \]

"the dispersion relation"

- Note: higher wavenumbers decay faster.
\( \kappa \), the wave number is determined by matching the boundary conditions.

Let's consider the initial-boundary value problem:

\[
U_t = \alpha^2 U_{xx}
\]

the diffusion equation.

\[0 < x < l, \ t > 0\]

Initial condition:

\[U(x,0) = f(x), \ 0 < x < l\]

Different boundary conditions:

1) Dirichlet problem:

\[U(0,t) = 0 = U(l,t)\]

2) Neumann problem:

\[U_x(0,t) = 0 = U_x(l,t)\]

\[
\sin \left( \frac{n \pi x}{L} \right) \quad n = 1, 2, 3, \ldots
\]

steady-state solution
3) mixed boundary conditions:

\( u(0, t) = 0 = u(L, t) \)

Solution to the Dirichlet problem by separation of variables:

\[ u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \]

\[ u(0, t) = 0 = u(L, t) \quad \text{Dirichlet BC} \]

\[ u(x, 0) = f(x) \quad \text{I. C.} \]

Guess a solution:

\[ u(x, t) = X(x) \cdot T(t) \]
\[ u_t = X(x) \cdot \dot{T}(t) \]
\[ u_{xx} = X''(x) \cdot T(t) \]
So:
\[ X(x) \cdot \ddot{T}(t) = \alpha^2 X(x) \cdot T(t) \]
Divide by \( \alpha^2 X T \):
\[ \frac{\ddot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \text{Constant} = \lambda \]
\[ \text{the separation constant} \]
This will give 2 ODEs.

\[ T(t) : \quad \ddot{T}(t) = \lambda \alpha^2 T(t) \]
\[ T(t) = C e^{\lambda \alpha^2 t} \]

\[ X(x) : \quad \dddot{X}(x) = \lambda \dddot{X}(x) \quad \text{an eigenvalue problem} \]
\[ X(0) = 0 = X(L) \]

An obvious solution is \( X(0) = 0 \), this is a trivial solution. Can we find a non-trivial solution?
The answer to this question depends on values of $\lambda \to$ an eigenvalue problem.

1) if $\lambda > 0$:

$$\lambda = \mu^2$$

$$\bar{X}'' - \mu^2 \bar{X} = 0 , \quad \bar{X} = e^{r x}$$

$$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r = \pm \mu$$

$$\cosh(\mu x) = \frac{e^{\mu x} + e^{-\mu x}}{2}$$

$$\sinh(\mu x) = \frac{e^{\mu x} - e^{-\mu x}}{2}$$

$$\bar{X}_+ = C_1 e^{-\mu x} + C_2 e^{\mu x}$$

or:

$$\bar{X}_+ = A \sinh(\mu x) + B \cosh(\mu x)$$

$$\bar{X}(0) = 0 \quad \Rightarrow \quad B = 0$$

$$\bar{X}(L) = 0 = A \sinh(\mu L) = 0 \quad \Rightarrow \quad A = 0$$

trivial solution

2) if $\lambda = 0$:

$$\lambda = 0 \quad \Rightarrow \quad \bar{X}'' = 0 , \quad \bar{X}' = A$$

$$\bar{X} = A x + B$$
$X(0) = B = 0, \quad X(L) = AL = 0 \rightarrow A = 0$

This option also gives \( X = 0 \), trivial solution

\( \lambda < 0 \): \( \lambda = -\mu^2 \) \( \Rightarrow \) \( X'' + \mu^2 X = 0 \)

\( X = e^{rx} \rightarrow r^2 + \mu^2 = 0 \)

\( r = \pm i \mu \)

\( X = A \sin(\mu x) + B \cos(\mu x) \)

\( X(0) = X(L) = 0 \rightarrow B = 0 \)

\( X(L) = 0 \rightarrow A \sin(\mu L) = 0 \)

if \( A = 0 \) \( \Rightarrow \) we get the trivial solution again.

So, \( A \neq 0 \), \( \sin(\mu L) = 0 \)

or: \( \mu L = n \pi, \quad n = 1, 2, 3, \ldots \)

\( \lambda_n = \left( \frac{\mu}{\sqrt{n}} \right)^2 = \left( \frac{n \pi}{L} \right)^2 \), \( n = 1, 2, 3, \ldots \) are eigenvalues

\( X_n(x) = \sin_\mu x = \sin \left( \frac{n \pi x}{L} \right) \) are eigenfunctions.
Shape of eigenfunctions:

\[
\begin{align*}
  n=1 & : \quad \sin \left( \frac{n\pi x}{L} \right) \\
  n=2 & : \quad \sin \left( \frac{2n\pi x}{L} \right) \\
  n=3 & : \quad \sin \left( \frac{3n\pi x}{L} \right)
\end{align*}
\]

So, the solutions will have the form:

\[
U_n(x, t) = e^{-\alpha \left( \frac{n\pi x}{L} \right)^2 t} \sin \left( \frac{n\pi x}{L} \right)
\]

\(n = 1, 2, 3, \ldots\)

A linear combination of these gives the general solution:

\[
U(x, t) = \sum_{n=1}^{\infty} b_n e^{-\alpha \left( \frac{n\pi L}{L} \right)^2 t} \sin \left( \frac{n\pi x}{L} \right)
\]

How can we find the coefficients \(b_n\)?

From the initial condition: \(U(x, 0) = f(x)\)
\[ u(x,0) = f(x) = \sum_{n=0}^{\infty} b_n \sin\left( \frac{n\pi x}{L} \right) \]

a Fourier series.