Math 257/316 Assignment 4, 2019
Due Friday February 8 IN CLASS

Problem 1 (Do not submit): Separation of Variables.
(a) Determine whether the method of separation of variables can be used to replace the following PDE’s by a pair of ODE’s. If so, find the equations.
(i) $x^2 u_{xx} = tu_t$. (ii) $u_{xx} + (x + y)u_{yy} = 0$.
(b) Consider the 2D heat diffusion equation

$$u_t = \alpha^2 (u_{xx} + u_{yy}).$$

Let the solution have the form $u(x,y,t) = X(x)Y(y)T(t)$. Find the ODEs that are satisfied by $X(x)$, $Y(y)$ and $T(t)$.

Problem 2 (Submit a, b, c, and d): Eigenvalue Problems. Find all eigenvalues and corresponding eigenfunctions for the following problems
(a) $y'' + \lambda y = 0, \quad (0 < x < 1), \quad y(0) = 0, \quad y(1) = 0.$
(b) $y'' + 2y' + \lambda y = 0, \quad (0 < x < \pi), \quad y(0) = 0, \quad y(\pi) = 0.$
(c) $x^2 y'' + xy' + \lambda y = 0, (1 < x < 2), \quad y(1) = 0 = y'(2)$
(d) $y'' + \lambda y = 0 \quad (0 < x < \pi), \quad y'(0) = 0, \quad y'(\pi) = 0.$
(e) $y'' + \lambda y = 0, \quad (0 < x < 2), \quad y(0) = 0, \quad y(2) = 0.$
(f) $x^2 y'' + 3xy' + \lambda y = 0, (1 < x < 3), \quad y(1) = 0 = y(3).$ Only consider the case $\lambda \geq 1$.

Problem 3 (Submit): Finite Difference Approximations.
(a) Use Taylor’s series expansions about the point $x$ for $f(x - \Delta x)$, and $f(x - 2\Delta x)$ to find a backward finite difference approximation for $f'(x)$ that has a second order accuracy.
(b) Consider the PDE for the convection-diffusion problem

$$u_t + cu_x = Du_{xx},$$

Use finite difference approximations for the derivatives in this equation to find a discrete form of the convection-diffusion equation. Explain how you would find $u(x, t + \Delta t)$ at each time step and discuss the order of accuracy of your numerical scheme in time and in space.

Problem 4 (Submit): The heat equation. Consider the following boundary value problem for the heat equation

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < \pi, \quad t > 0, \quad \alpha^2 = 2.0$$

**BC**: $u_x(0, t) = 0, \quad u_x(\pi, t) = 0$

**IC**: $u(x, 0) = \cos(x/2)$
(a) Use the method of separation of variables to solve the above boundary value problem.

(b) The Matlab script attached to this assignment solves the discrete heat equation

\[
\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \alpha^2 \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2},
\]

for user specified boundary conditions and initial values. Modify this code and run it in Matlab to find the numerical solution of the problem. In a plot compare the solution you found in (a) to the numerical solution obtained from the code at \( t = 0.2 \).

(c) Find the steady-state solution by setting \( u_t = 0 \) in the heat equation. Compare this solution to the steady-state solution you get from the Matlab code. (Note: You have to run the code for a large enough time interval to make sure a steady state is reached.) Explain why you get this steady-state solution.

**Problem 5 (Do not submit): Modes of vibration of a simply supported beam.**

The vibrations of an elastic bar of length \( L \) are governed by the fourth order partial differential equation

\[
\alpha^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0.
\]

(a) Assuming a sinusoidal time variation of the solution of the form \( y(x, t) = \exp(i\omega t)w(x) \) show that the spatial component \( w(x) \) satisfies

\[
w''' - \left( \frac{\omega}{\alpha} \right)^2 w = 0 \quad (3)
\]

(b) Show that the general solution of the 4th order ODE (3) is of the form

\[
w(x) = A \cos \mu x + B \sin \mu x + C \cosh \mu x + D \sinh \mu x
\]

(c) If the beam is simply supported then the endpoints are subject to the boundary conditions

\[
\begin{align*}
w(0) &= 0 = w(L) \\
w''(0) &= 0 = w''(L)
\end{align*} \quad (4)
\]

Determine the eigenvalues and eigenfunctions associated with the eigenvalue problem comprising (3) subject to the boundary conditions (4).

**Hint:** First use the conditions \( w(0) = 0 = w''(0) \) to show that \( A = C = 0 \). Use the remaining two conditions \( w(L) = 0 = w''(L) \) to arrive at a \( 2 \times 2 \) system of equations for the remaining constants \( B \) and \( D \). Using the condition for a non-trivial solution to this system determine the resulting eigenvalues and eigenfunctions.

(d) Now determine an expression for the lowest frequency of vibration of the beam, i.e. the value of \( \omega \) associated with the smallest eigenvalue.