The one-dimensional wave equation:

\[ u_{tt} = c^2 u_{xx} \]

To solve this PDE we need two initial conditions (second order derivative in time) and two boundary conditions (second order time derivative in space), e.g.:

\[
\begin{align*}
  u(x,0) &= f(x) & \{ & \text{I.C.'s} \\
  u_t(x,0) &= g(x) \\
  u(0,t) &= 0 & \{ & \text{B.C.'s} \\
  u(L,t) &= 0
\end{align*}
\]

Physical models (review):

**Elastic bar:**

\[
\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}
\]

\[
\frac{\partial \sigma}{\partial x} = E \frac{\partial^2 u}{\partial x^2}
\]

Balance of linear momentum

\[
\frac{\partial u}{\partial x} = \varepsilon 
\]

Constitutive law

\[ E : \text{Young's modulus} \]

\[ \varepsilon = \frac{\partial u}{\partial x} : \text{strain} \]
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad c = \sqrt{\frac{E}{\rho}} \]

Check units: \[ \left[ c^2 \right] = \left[ \left( \frac{E}{\rho} \right) \right] = \frac{kg \cdot m}{s^2} \cdot \frac{1}{m^2} \]
\[ = \frac{kg}{m^3} \]
\[ = \frac{m^2}{s^2} \]

Vibrating string:

No motion in the horizontal direction:

\[ T_1 \cos \Theta_1 = T_2 \cos \Theta_2 = T \]

Write Newton's second law for the vertical motion:

\[ T_2 \sin \Theta_2 - T_1 \sin \Theta_1 = \rho \Delta x \frac{\partial^2 u}{\partial t^2} \quad [\rho] = \frac{kg}{m} \]

Divide both sides by \( T \):

\[ \frac{\sin \Theta_2}{\cos \Theta_2} - \frac{\sin \Theta_1}{\cos \Theta_1} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} \]
\[ \tan(\theta_1) = \frac{\partial u}{\partial x} \bigg|_x, \quad \tan(\theta_2) = \frac{\partial u}{\partial x} \bigg|_{x+\Delta x} \]

\[ \Rightarrow \quad \frac{1}{\Delta x} \left( \frac{\partial u}{\partial x} \bigg|_{x+\Delta x} - \frac{\partial u}{\partial x} \bigg|_x \right) = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2} \]

Let \( \Delta x \to 0 \):

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2} \]

or:

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c = \sqrt{\frac{T}{\rho}} \]

Check the units:

\[ [\frac{T}{\rho}] = \frac{\text{kg} \cdot \text{m/s}^2}{\text{kg/m}} = \text{m/s}^2 \]

Shallow water waves:

\[ c = \sqrt{gh} \]

\[ [gh] = \text{m/s}^2 \cdot \text{m} = \text{m}^2/\text{s}^2 \]

Example: Tsunami wave: Ocean depth \( \approx 10 \text{ m} \)

\( g \approx 10 \text{ m/s}^2 \)
Tsunami wave speed:

\[ c = \sqrt{10 \cdot 10^3} = 100 \text{ m/s} = 360 \text{ km/hr} \]

The 1D wave equation can be decomposed into a right moving and a left moving wave:

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = 0 \]

Remember that we can look for an exponential solution:

\[ u(x,t) = e^{ikx+6t} \]

\[ u_{tt} = c^2 u_{xx} \]

\[ \rightarrow \quad 6^2 e^{ikx+6t} = -k^2 c^2 e^{ikx+6t} \]

\[ 6^2 = -k^2 c^2 \quad \text{or} \quad 6 = \pm ikc \]

So, the guess solution has the form:

\[ u(x,t) = e^{ik(x+ct)} \]

This hints that \( x+ct \) has a physical significance.
Calilean transformation:

By using the Calilean transformation: \( x' = x \pm ct \), corresponding to a frame of reference moving to the right at speed \( c \), you find a solution to the wave equation:

\[
U(x_1t) = G(x \pm ct) \\
U_{xx} = G'', \quad U_{tt} = (\pm c)^2 G'' \\
\Rightarrow \quad U_{tt} - c^2 U_{xx} = c^2 G'' - c^2 G'' = 0
\]

D'Alambert's solution to the wave equation:

Consider the 1D wave equation:

\[
U_{tt} = c^2 U_{xx}
\]

subject to initial conditions:

\[
U(x,0) = f(x), \quad U_t(x,0) = g(x)
\]

Let the solution be the sum of a right moving and a left moving wave:

\[
U(x,t) = F(x-ct) + G(x+ct) = \overleftarrow{\overrightarrow{c}} + \overrightarrow{\overleftarrow{c}}
\]
Apply the I.C.'s:

\[ p(x) = u(x, 0) = F(x) + C(x) \]

\[ g(x) = u_t(x, 0) = -C F'(x) + C_G'(x) \]

\[ \int g(s) ds + A \]

\[ C F(x) + C G(x) = C \cdot F(x) \]

Solve for \( F(x) \) and \( G(x) \):

\[ 1 + 2 \Rightarrow G(x) = \frac{1}{2} F(x) + \frac{1}{2c} \int_0^x g(s) ds + \frac{A}{2c} \]

\[ 2 - 1 \Rightarrow F(x) = \frac{1}{2} F(x) - \frac{1}{2c} \int_0^x g(s) ds - \frac{A}{2c} \]

So, the time-dependent solution is

\[ u(x, t) = F(x - ct) + G(x + ct) \]

\[ = \frac{1}{2} \left\{ F(x - ct) + F(x + ct) \right\} + \frac{1}{2c} \left\{ \int_0^{x-ct} g(s) ds + \int_0^{x+ct} g(s) ds \right\} - \frac{A}{2c} + \frac{A}{2c} \]

or:

\[ u(x, t) = \frac{1}{2} \left\{ F(x - ct) + F(x + ct) \right\} + \frac{1}{2c} \int_0^{x+ct} g(s) ds \]

D'Alembert's Solution to wave eqn. on \((-\infty, \infty)\)
Solving the wave equation using separation of variables:

Consider the vibrating string problem, now with a finite length:

\[ u_{tt} = c^2 u_{xx} \quad 0 < x < L \]

B.C.: \( u(0,t) = 0 = u(L,t) \)

I.C.: \( u(x,0) = f(x) \), \( u_t(x,0) = g(x) \)

Let: \( u(x,t) = X(x) \cdot T(t) \)

Substitute into the PDE: \( X(x) \cdot T(t) = c^2 X''(x) \cdot T(t) \)

Divide by \( c^2 X(x) \cdot T(t) \):

\[ \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\mu^2 \neq 0 \quad \text{(positive and zero constants give trivial solution)} \]

\[ X'' + \mu^2 X = 0 \]

\[ X(0) = 0 = X(L) \]

\[ X(x) = A \cos(\mu x) + B \sin(\mu x) \]

\[ X(0) = 0 \rightarrow A = 0 \]

\[ X(L) = 0 \rightarrow B \sin(\mu L) = 0 \]

\[ B \neq 0 \rightarrow \mu L = n \pi \]

\[ \mu = \frac{n \pi}{L} \quad n = 1, 2, 3, \ldots \]
\[ T + \mu^2 c^2 T = 0 \quad \rightarrow \quad T(t) = A_n \cos(\mu_n c t) + B_n \sin(\mu_n c t) \]

\[ U(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos(\mu_n c t) + B_n \sin(\mu_n c t) \right\} \sin(\mu_n x) \]

Take the time derivative of the solution:

\[ U_t(x,t) = \sum_{n=1}^{\infty} \left\{ -\mu_n^2 c A_n \sin(\mu_n c t) + \mu_n^2 c B_n \cos(\mu_n c t) \right\} \sin(\mu_n x) \]

Apply the I.C.'s:

\[ U(x,0) = \sum_{n=1}^{\infty} A_n \sin(\mu_n x) = f(x) \]

So, \( A_n \) are the Fourier sine series coefficients of \( f(x) \):

\[ A_n = \frac{2}{L} \int_{0}^{L} f(x) \sin(\mu_n x) \, dx = b_n \]

\[ U_t(x,0) = \sum_{n=1}^{\infty} B_n \mu_n^2 c \cos(\mu_n x) = g(x) \]

\[ B_n = \frac{2}{\mu_n^2 c L} \int_{0}^{L} g(x) \sin(\mu_n x) \, dx \]

\[ = \frac{\frac{g}{b_n}}{\mu_n c} \]
So, the solution is:

\[ u(x,t) = \sum_{n=1}^{\infty} \left( b_n \cos\left( \frac{\mu_n ct}{c} \right) + \frac{g}{\mu_n c} \sin\left( \frac{\mu_n ct}{c} \right) \right) \sin\left( \frac{\mu_n x}{c} \right) \]

Now, how does this compare to D'Alembert's Solution? Use trigonometric identities to expand \( \cos\left( \frac{\mu_n ct}{c} \right) \), \( \sin\left( \frac{\mu_n x}{c} \right) \)

and \( \sin\left( \frac{\mu_n ct}{c} \right) \), \( \sin\left( \frac{\mu_n x}{c} \right) \).

Identities:

1. \( \sin (A \pm B) = \sin A \cos B \pm \cos A \sin B \)
2. \( \cos (A \pm B) = \cos A \cos B \mp \sin A \sin B \)

\[ \frac{1}{2} \left\{ \sin (A + B) + \sin (A - B) \right\} \quad (1) \]
\[ \frac{1}{2} \left\{ \cos (A - B) - \cos (A + B) \right\} \quad (2) \]

So, \( u(x,t) \) becomes:

\[ u(x,t) = \sum_{n=1}^{\infty} \left( \frac{b_n}{2} \left[ \sin\left( \frac{\mu_n (x+ct)}{c} \right) + \sin\left( \frac{\mu_n (x-ct)}{c} \right) \right] + \frac{g}{2 \mu_n c} \left[ \cos\left( \frac{\mu_n (x-ct)}{c} \right) - \cos\left( \frac{\mu_n (x+ct)}{c} \right) \right] \right) \]
Call the two sums \( U_f \) and \( U_g \), so that:

\[
U(x,t) = U_f + U_g
\]

Also, remember that:

\[
0 \leq x < L \\
\begin{align*}
\hat{f}(x) &= \sum_{n=1}^{\infty} b_n \sin\left( \frac{\mu_n x}{L} \right) \\
\hat{g}(x) &= \sum_{n=1}^{\infty} b_n \sin\left( \frac{\mu_n x}{L} \right)
\end{align*}
\]

So:

\[
U = \sum_{n=1}^{\infty} \frac{b_n}{2} \left[ \sin\left( \frac{\mu_n (x+ct)}{L} \right) + \sin\left( \frac{\mu_n (x-ct)}{L} \right) \right]
\]

\[
= \frac{1}{2} \left[ \hat{f}(x+ct) + \hat{f}(x-ct) \right]
\]

Also, note that from \((*)\) you have:

\[
\int_{0}^{x} g(s) \, ds = \sum_{n=1}^{\infty} \int_{0}^{x} b_n \sin\left( \frac{\mu_n x}{L} \right) \, dx
\]

\[
= \sum_{n=1}^{\infty} \frac{b_n}{\mu_n} \left( -\cos\left( \frac{\mu_n x}{L} \right) \right) + A
\]

\[\uparrow\]

a constant
So,

\[ u = \sum_{n=1}^{\infty} \frac{b_n}{2^{\frac{M_n}{c}}} \cos \left( \frac{M_n}{c}(x-ct) \right) - \sum_{n=1}^{\infty} \frac{b_n}{2^{\frac{M_n}{c}}} \cos \left( \frac{M_n}{c}(x+ct) \right) \]

\[ = \frac{1}{2c} \int_{x-ct}^{0} g(s) \, ds + A + \frac{1}{2c} \int_{0}^{x+ct} g(s) \, ds - A \]

\[ = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds \]

So, you can conclude that separation of variables and D'Alembert's solutions are the same:

\[ u(x,t) = u + u \]

\[ = \sum_{n=1}^{\infty} \left\{ b_n \cos \left( \frac{M_n}{c}ct \right) + \frac{b_n}{M_n} \sin \left( \frac{M_n}{c}ct \right) \right\} \sin \left( \frac{M_n}{c}x \right) \]

\[ = \frac{1}{2} \left\{ f(x+ct) + f(x-ct) \right\} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds \]

Note: \( f \), \( g \) are odd extensions of \( f \) and \( g \)