Laplace's equation on rectangular domains with Neumann B.C.'s and semi-infinite domains:

Example: Neumann problem on a rectangle

Consider the domain with flux boundary conditions

\[ \Delta u = u_{xx} + u_{yy} = 0 \]

BC: \[ u_x(0, y) = 0 = u_x(a, y) \]
\[ u_y(x, 0) = 0, \quad u_y(x, b) = f(x) \]
\[ u_y(x, 10) = f(x) \]

opposite homogeneous B.C.'s are in \( x \) direction \( \rightarrow \) an eigenvalue problem in \( x \)

Separation of variables:

\[ u(x, y) = X(x) \cdot Y(y) \]
\[ X''Y + Y''X = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda = -\mu^2 \]

\[ X'' + \mu^2 X = 0 \]
\[ X'(0) = 0 = X'(a) \]
\[ \mu_n \in \{ 0, \frac{n\pi}{a} \} \quad n = 1, 2, \ldots \]
\[ X_n \in \{ 1, \cos(\frac{n\pi x}{a}) \} \]
Note: \( \lambda > 0 \) will give a trivial solution \( X = 0 \).

\[
Y'' - \mu^2 Y = 0
\]

\( \mu \neq 0 \):

\[
Y = A \cosh(\mu y) + B \sinh(\mu y)
\]

\[
Y' = A \mu \sinh(\mu y) + B \mu \cosh(\mu y)
\]

\[
Y'(b) = 0 \Rightarrow A \mu \sinh(\mu b) + B \mu \cosh(\mu b) = 0
\]

\[
B = -A \frac{\sinh(\mu b)}{\cosh(\mu b)} = \tanh(\mu b)
\]

\[
Y(y) = A \cosh(\mu y) - A \frac{\sinh(\mu b)}{\cosh(\mu b)} \cdot \sinh(\mu y)
\]

\[
= A \left\{ \frac{\cosh(\mu y) \cosh(\mu b) - \sinh(\mu b) \sinh(\mu y)}{\cosh(\mu b)} \right\}
\]

\[
= \frac{A}{\cosh(\mu b)} \quad \text{Cosh} \left( \mu (y-b) \right) = D \cosh(\mu (y-b))
\]

\( \mu = 0 \):

\[
Y'' = 0, \quad Y' = C, \quad Y = Cy + D
\]

\[
Y'(b) = C = 0, \quad Y_0 = D_0 \cdot 1, \quad M_0 = 0
\]
So, the general solution is:

\[ u(x, y) = D_0 + \sum_{n=1}^{\infty} D_n \cosh(\mu_n(y-b)) \cos(\mu_n x) \]

where \( \mu_n = \frac{n\pi}{a} \)

We need to apply the inhomogeneous B.C.,

\[ u_y(x, 0) = f(x) \]

To find coefficients \( D_n \):

Take partial \( y \) derivative of \( u(x, y) \):

\[ u_y(x, y) = 0 + \sum_{n=1}^{\infty} D_n \mu_n \sinh(\mu_n(y-b)) \cos(\mu_n x) \]

\[ u_y(x, 0) = f(x) = 0 + \sum_{n=1}^{\infty} -D_n \mu_n \sinh(\mu_n b) \cos(\mu_n x) \]

This is similar to the Fourier cosine expansion of \( f(x) \):

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\mu_n x) \]

This problem has a solution only if:

\[ a_0 = \frac{2}{a} \int_{0}^{a} f(x) \, dx = 0 \]
From a physical point of view, a steady-state solution of the 2D heat conduction problem, i.e.

\[ u_t = 0 = u_{xx} + u_{yy} \]

is only possible if the influx of heat has a zero average:

\[ \int_0^a F(x) \, dx = 0. \]

If this is true, then \( D_n \) coeffs are found from:

\[ -D_n g_n \sinh(g_n b) = a_n^f = \frac{2}{a} \int_0^a F(x) \cos(g_n x) \, dx \]

\[ \Rightarrow D_n = \frac{a_n^f}{g_n \sinh(-g_n b)} \]

Substituting this into \( u(x,y) \) solution:

\[ u(x,y) = D_0 + \sum_{n=1}^{\infty} \frac{a_n^f}{g_n \sinh(-g_n b)} \frac{\cosh(g_n (y-b))}{\cos(g_n x)} \]

\[ g_n = \frac{n \pi c}{a} \quad n=1, 2, \ldots \]

Still an arbitrary constant.
\( u(x,y) \) is said to be known up to an arbitrary constant.

- If \( u_\infty (x,y) \) is the steady-state solution of the heat conduction problem \( u_t = u_{xx} + u_{yy} \), with initial condition: \( u(x, y, 0) = u_0 (x,y) \):

\[
\frac{\partial}{\partial t} \int \int_0^b u \, dx \, dy = 0
\]

\[
\int \int_0^b u \, dx \, dy = \text{constant} = \int \int_0^b u_0 (x,y) \, dx \, dy
\]

\[
D_0 \cdot a \cdot b + \sum_{n=1}^{\infty} \frac{a_n}{\mu_n \sinh(\mu_n b)} \cdot \int_0^b \cosh(\mu_n (y-b)) \, dy \cdot \int_0^a \cos(n \pi x / a) \, dx
\]

\[
= \int \int_0^b u_0 (x,y) \, dx \, dy
\]

\[
D_0 = \frac{1}{ab} \int \int_0^b u_0 (x,y) \, dx \, dy
\]
Example: Laplace's equation on a semi-infinite strip with mixed, inhomogeneous B.C.'s

\[ \Delta u = u_{xx} + u_{yy} = 0 \quad 0 < x < a \quad 0 < y < \infty \]

\[ u(x,0) = \phi_0, \quad u(x,a) = \phi_1 \]

\[ u(x,y) \rightarrow \phi_0(x-a) + \phi_1 \quad \text{as} \quad y \rightarrow \infty \]

\[ \Delta u = 0 \]

\[ u(x,0) = \phi_0 \quad u(a, y) = \phi_1 \]

\[ u(x,a) = \phi_1 \quad u(x,0) = \phi_0 \]

First, remove the inhomogeneous B.C.'s and transform the problem into a homogeneous B.C. problem:

Let \( w(x) = Ax + B \) satisfy the inhomog. B.C.:

\[ w(x) = Ax + B \quad , \quad w_x = A \]

\[ w_x(0) = \phi_0 \quad \Rightarrow A = \phi_0 \]

\[ w(a) = \phi_1 \quad \Rightarrow \phi_0 a + B = \phi_1 \quad \Rightarrow B = \phi_1 - \phi_0 a \]

\[ w(x) = \phi_0 x + \phi_1 - \phi_0 a = \phi_0 (x-a) + \phi_1 \]
Now decompose the problem into \( w \) and \( v \):

Let \( u(x,t) = w(x) + v(x,t) \)

\[
\begin{align*}
    u_{xx} + u_{yy} &= \frac{w_{xx}}{0} + \frac{v_{xx}}{0} + \frac{u_{yy}}{0} + v_{yy} \\
    &= v_{xx} + v_{yy} = 0 \quad \text{PDE}
\end{align*}
\]

\[
\begin{align*}
    u(x,0) &= f(x) = w(x) + v(x,0) \\
    \implies v(x,0) &= f(x) - w(x)
\end{align*}
\]

\[
\begin{align*}
    u(x,\infty) &= \phi_0(x-a) + \phi_1 = \phi_0(x-a) + \phi + v(x,\infty) \\
    \implies v(x,\infty) &= 0
\end{align*}
\]

\[
\begin{align*}
    u_x(0,y) &= \phi_0 = w_x(0) + v_x(0,y) \\
    &= \phi_0 + v_x(0,y) \\
    \implies v_x(0,y) &= 0
\end{align*}
\]

\[
\begin{align*}
    u(a,y) &= \phi_1 = w(a) + v(a,y) \\
    &= \phi_1 + v(a,y) \\
    \implies v(a,y) &= 0
\end{align*}
\]
So, the B.V.P. for \( V(x,t) \) is:

\[
V_{xx} + V_{yy} = 0 \quad 0 < x < a, \quad 0 < y < \infty
\]

\[
V(x,0) = f(x) - w(x)
\]

\[
V(x,y) \to 0 \quad \text{as} \quad y \to \infty
\]

\[
V_x(0,y) = 0 = V(a,y)
\]

Separation of variables:

Let: \( V(x,y) = X(x) \cdot Y(y) \)

\[
V_{xx} + V_{yy} = X'' \cdot Y + Y'' \cdot X = 0
\]

\[
\frac{X''}{X} = -\frac{Y''}{Y} = \lambda = -\mu^2
\]

(An eigenvalue problem in \( x \))

\[
\begin{align*}
X'' + \mu^2 X &= 0 \\
X'(0) &= 0 = X(a)
\end{align*}
\]

\[
\mu_n = \frac{(2n-1)\pi}{2a}, \quad n = 1, 2, \ldots
\]

\[
X_n = \cos(\mu_n x)
\]
\[ y'' - \mu^2 y = 0 \quad \rightarrow \quad y = C \cosh(\mu y) + D \sinh(\mu y) \]

To apply the \( y \to \infty \) it is easier to write \( y \) as exponentials:

\[ y = A e^{\mu y} + B e^{-\mu y} \]

\( y \to 0 \) as \( y \to \infty \) : \( A = 0 \)

So,

\[ y_n(y) = e^{-\mu_n y} \]

\[ V_n(x, y) = e^{-\mu_n y} \cos(\mu_n x), \quad \mu_n = \frac{(2n-1)\pi}{2a}, \quad n = 1, 2, 3, \ldots \]

\[ V(x, y) = \sum_{n=1}^{\infty} B_n e^{-\mu_n y} \cos(\mu_n x) \]

Apply the inhomogeneous B.C.: \( V(x, 0) = f(x) - w(x) \)

to find \( B_n \) coeffs:

\[ V(x, 0) = f(x) - w(x) = \sum_{n=1}^{\infty} B_n \cos(\mu_n x) \]
So, \( B_n \) are Fourier cos coefficients of 

\[ f(x) - \omega(x) : \]

\[ B_n = \frac{2}{a} \int_{0}^{a} [f(x) - \omega(x)] \cos \left( \frac{(2n-1)\pi x}{2a} \right) \, dx \]

The final solution is:

\[ u(x,y) = \phi_0 (x-a) + \phi_1 + \sum_{n=1}^{\infty} B_n e^{-\left( \frac{(2n-1)\pi y}{2a} \right)} \cos \left( \frac{(2n-1)\pi x}{2a} \right) \]