Laplace's equation on polar coordinates:

\[ \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \]

Separation of variables:

\[ u(r, \theta) = \Theta(\theta) \cdot R(r) \]

\[ \Delta u = R'' \Theta + \frac{1}{r} R' \Theta + \frac{1}{r^2} R \Theta'' = 0 \]

Multiply by \( \frac{r^2}{R \Theta} \):

\[ \frac{r^2 R''}{R} + \frac{r R'}{R} = -\frac{\Theta''}{\Theta} = \lambda \]

The separation constant should be:

\( \mu^2 \): For eigenvalue problems in \( \Theta \), i.e.

\[ \Theta(0) = \Theta(\alpha) = 0 \]

\( -\mu^2 \): For eigenvalue problems in \( R \), i.e.

\[ R(r_1) = R(r_2) = 0 \]
Eigenvalue problems in θ:

\[ \lambda = \mu^2 \]

\( \mu \neq 0 \):
\[ \theta'' + \mu^2 \theta = 0 \rightarrow \theta = A \cos(\mu \theta) + B \sin(\mu \theta) \]

\( \mu = 0 \):
\[ \theta'' = 0 \rightarrow \theta = A \theta + B \]

Different boundary conditions and their eigenfunctions:

(I) Dirichlet:
\[ \theta(0) = 0 = \theta(\alpha) \]

\[ \mu_n = \frac{n \pi c}{\alpha} \]

\[ \theta_n = \sin(\mu_n \theta) \]

\[ n = 1, 2, 3, \ldots \]

(II) Neumann:
\[ \theta'(0) = 0 = \theta'(\alpha) \]

\[ \mu_n \in \left\{ 0, \frac{n \pi c}{\alpha} \right\} \]

\[ \theta_n \in \left\{ 1, \cos(\mu_n \theta) \right\} \]
(III) Periodic:
\[ \Theta(-\pi) = \Theta(\pi) \]
\[ \Theta'(-\pi) = \Theta'(\pi) \]

\[ \mu_n \in \left\{ 0, \frac{n\pi}{\alpha} \right\} \]
\[ \Theta_n \in \{1, \cos(n\alpha), \sin(n\alpha)\} \]
\[ n = 1, 2, \ldots \]

(IV) Mixed B.C., Type I:
\[ \Theta(0) = 0 = \Theta'(\alpha) \]
\[ \mu_n = \frac{(2n-1)\pi}{2\alpha}, \quad n = 1, 2, 3, \ldots \]
\[ \Theta_n = \sin(\mu_n \theta) \]

(V) Mixed B.C., Type II:
\[ \Theta'(0) = 0 = \Theta(\alpha) \]
\[ \mu_n = \frac{(2n+1)\pi}{2\alpha}, \quad n = 1, 2, 3, \ldots \]
\[ \Theta_n = \cos(\mu_n \theta) \]
Solutions to the R ODE:

\[ \mu \neq 0 : \quad r^2 R'' + rR' - \mu^2 R = 0 \]

This is a Cauchy-Euler equation:

**Guess:** \( R = r^\gamma, \quad R' = \gamma r^{\gamma-1}, \quad R'' = \gamma(\gamma-1) r^{\gamma-2} \)

\[ r^\gamma \left[ \gamma(\gamma-1) + \gamma - \mu^2 \right] = 0 \]

\[ \gamma^2 - \gamma + \gamma - \mu^2 = 0 \quad \rightarrow \quad \gamma = \pm \mu \]

\[ R(r) = cr^\mu + dr^{-\mu} \]

**\( \mu = 0 : \quad r^2 R'' + rR' = 0 \)**

C-E equation \( \rightarrow \) **Guess:** \( R = r^\gamma \)

\[ \gamma(\gamma-1) + \gamma = 0 \quad \rightarrow \quad \gamma = 0, \quad \gamma = 0, 0 \text{ repeated roots} \]

\[ R(r) = c_1 + D \ln r \]

Therefore, the most general solution is:
\[ u(r, \theta) = \left\{ A_0 + \alpha_0 \ln r \right\} \cdot 1 \]

+ \sum_{n=1}^{8} \left\{ A_n r^{n} + \alpha_n r^{-n} \right\} \cos (\lambda_n \theta) \]

+ \sum_{n=1}^{8} \left\{ B_n r^{n} + \beta_n r^{-n} \right\} \sin (\lambda_n \theta) \]

\text{(\ast)}

where \( \lambda_n \) is an eigenvalue from one of problems (I) to (IV).

\text{Note:}

- If the problem includes the origin, and \( |u| \) is finite as \( r \to 0 \), then:

\[ \alpha_0 = 0, \ \alpha_n = 0, \ \beta_n = 0 \]

- If the problem has infinite domains, and \( |u| \) is finite as \( r \to \infty \), then:

\[ A_n = 0, \ B_n = 0 \]
Example: A crack problem:

\[ \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0 \]

B.C.'s: \[ u(r, 0) = u_0(r, 0) \]
\[ u \to 0 \text{ as } r \to \infty \]
\[ u(a, \theta) = f(\theta) \]

An eigenvalue problem in \( \theta \):

\[ \frac{d^2 \theta}{d \theta^2} + \mu \theta = 0 \]

with mixed type I B.C.'s:

\[ \mu_n = \frac{(2n-1) \pi}{2 \pi} = \frac{2n-1}{2} , \quad n = 1, 2, 3, \ldots \]

\[ \Theta_n = \sin(\mu_n \theta) \]

Now, using the general solution in (*):

Mixed B.C.'s, so \( \mu = 0 \):

A_0 = 0, \quad \alpha_0 = 0

Mixed, type I B.C.:

\[ A_n = 0, \quad \alpha_n = 0 \]

(No cos eigenfunctions)

As \( r \to 0 \), \( u \to 0 \):

\[ \beta_n = 0 \]
So, the solution is:

\[ u(r, \theta) = \sum_{n=1}^{\infty} B_n r^{\mu_n} \sin(\mu_n \theta), \quad \mu_n = \frac{2n-1}{2} \]

\[ f(\theta) = u(a, \theta) = \sum_{n=1}^{\infty} B_n a^{\mu_n} \sin(\mu_n \theta) \]

\[ = \sum_{n=1}^{\infty} b_n \sin(\mu_n \theta) \]

a Fourier sin expansion of \( f(\theta) \)

\[ B_n a^{\mu_n} = b_n = \frac{2}{12} \int_0^\pi f(\theta) \sin(\mu_n \theta) \, d\theta \]

\[ B_n = \frac{b_n}{a^{\mu_n}} = \frac{2}{\mu_n} \int_0^\pi f(\theta) \sin(\mu_n \theta) \, d\theta \]

Substituting for \( B_n \) coefficients:

\[ u(r, \theta) = \sum_{n=1}^{\infty} b_n \left( \frac{r}{a} \right)^{\mu_n} \sin\left( \frac{(2n-1)}{2} \theta \right) \]
Now, assume:

\[ f(\theta) = \sin\left(\frac{\theta}{2}\right) \]

\[ a = 1 \]

or on the domain:

\[ b_n = \delta_n 1 \rightarrow u(r, \theta) = \left( \frac{r}{a} \right) \cdot \sin\left(\frac{\theta}{2}\right) \]