Math 257/316 Assignment 9, 2019  
Due date: April 1, 2019 IN CLASS

**Problem 1 (Submit):** Consider the static deflection of a plate that satisfies the Laplace’s equation and is subject to the following boundary conditions

\[ u_{xx} + u_{yy} = 0 \text{ for } 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2 \]

\[ u_x(3, y) = 0, \quad u_y(x, 0) = 0 \]

\[ u(x, 2) = f(x) = \begin{cases} 
  x & \text{if } 0 \leq x < 1 \\
  2 - x & \text{if } 1 \leq x < 3 
\end{cases} \]

\[ u(0, y) = f(y) = \begin{cases} 
  y & \text{if } 0 \leq y < 1 \\
  2 - y & \text{if } 1 \leq y < 2 
\end{cases} \]

The above boundary conditions state that the plate has zero slope on its \( x = 3 \) and \( y = 0 \) sides and undergoes triangular deformations on the \( x = 0 \) and \( y = 2 \) sides.

(a) Use the method of separation of variables to solve this boundary value problem.

(b) Verify that the solution you found satisfies all the boundary conditions.

(c) Use the Matlab code attached to this assignment to solve this problem numerically. Attach the plot of the shape of the plate to your assignment. Compare the value of the solution for \( u(3, 0) \) from the Matlab code and your analytical solution.

**Problem 2 (Submit)** A metal plate occupies a quarter-annular region \( \Omega \) with \( 0 < a \leq r \leq b \) and \( 0 \leq \theta \leq \pi/2 \). The horizontal face of the plate is insulated while the vertical face is kept at 2 degrees and the outer hoop is maintained at 2 degrees. The inner hoop is maintained at a temperature of \( \cos(2\theta) \). Determine the steady state temperature by solving the following BVP in \( \Omega \):

\[
\begin{cases}
  v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 & \text{in } \Omega \\
  v_\theta(r, 0) = 0 & \text{for } a < r < b \\
  v(r, \pi/2) = 2 & \text{for } a < r < b \\
  v(a, \theta) = \cos 2\theta & \text{for } 0 < \theta < \pi/2 \\
  v(b, \theta) = 2 & \text{for } 0 < \theta < \pi/2 
\end{cases}
\]

**Problem 3 (Do not submit):** Consider the BVP:

\[
\phi'' + 6\phi' + \lambda\phi = 0, \quad 0 < x < L \\
\phi(0) = 0 \\
\phi(L) = 0
\]

(a) Put this BVP into Sturm-Liouville form.

(b) Compute all eigenvalues and eigenfunctions.
(c) Show explicitly that the eigenfunctions are mutually orthogonal. (Don’t forget to include the weight function inside the integral.)

**Problem 4 (Do not submit):** Consider the eigenvalue problem

\[ x^2 y'' + xy' + \lambda y = 0 \]
\[ y(1) = 0 = y'(2) \]

a. Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions.
b. Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace’s equation on the quarter-annular region:

\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0, \quad 1 < r < 2, \quad 0 < \theta < \pi/2 \]
\[ u(r,0) = 0 \quad \text{and} \quad \frac{\partial u(r, \pi/2)}{\partial \theta} = f(r) \]
\[ u(1, \theta) = 0 \quad \text{and} \quad \frac{\partial u(2, \theta)}{\partial r} = 0 \]

**Problem 5 (Do not submit):**

Solve the following heat conduction problem:

\[ u_t = x^2 u_{xx} + 4x u_x \quad \text{for} \ x \in (1,2), \ t > 0 \]
\[ u(1,t) = 1 \quad u(2,t) = 1 \]
\[ u(x,0) = 1 - 5x^{-3/2} \]