Convergence of Fourier series:

Theorem: If $f$ and $f'$ are piecewise continuous functions on $[-L, L]$, and $f$ is periodic with period $2L$, then $f$ has a Fourier series, $S(x)$.

At points of discontinuity:

- $S(x)$ converges to $\frac{1}{2} \left[ f^+(x) + f^-(x) \right]$.
- Gibbs phenomenon observed in truncated series $S_N(x)$. 
Parseval's Theorem:

If function $f(x)$, periodic on $[-L, L]$, has a full-range Fourier series:

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right)$$

that converges to the function:

$$\lim_{N \to \infty} S_N(x) = f(x),$$

then the Fourier coefficients $a_n, b_n, a_0$ satisfy Parseval's identity:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{L} \int_{-L}^{L} f(x)^2 \, dx = E[f]$$

$E[f]$: the energy of function $f(x)$, with period $2L$.
Proof for Fourier sine series:

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \]

\[ \int_0^L f(x)^2 \, dx = \int_0^L \left( \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right) \right) \left( \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \right) \, dx \]

\[ = \sum_{m,n=1}^{\infty} b_m b_n \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \, dx \]

Orthogonality:

\[ \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases} \]

\[ = \frac{L}{2} \sum_{n=1}^{\infty} b_n^2 \]

Example:

\[ f(x) = x^2 \quad -\pi < x < \pi \quad 2L = 2\pi \rightarrow L = \pi \]

We found the Fourier expansion for this case:

\[ x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^2} \]
Now apply Parseval's theorem:

\[ \frac{1}{L} \int_{-L}^{L} f(x)^2 \, dx = \frac{1}{L} \int_{-L}^{L} x^4 \, dx \]

\[ = \frac{1}{L} \int_{-L}^{L} x^5 \, dx \bigg|_{-L}^{L} = \frac{2L^4}{5} \]  \hspace{1cm} (1)

\[ a_0^2 + \sum_{n=1}^{8} a_n^2 = \frac{1}{2} \left( \frac{2L^2}{3} \right)^2 + \sum_{n=1}^{8} \frac{4 \cdot (-1)^n}{n^4} \]

\[ = \frac{2L^4}{9} + 4 \sum_{n=1}^{8} \frac{1}{n^4} \]  \hspace{1cm} (2)

\[ (1) = (2) \]

\[ \frac{18 - 10}{45} L^2 = \frac{4}{9} \sum_{n=1}^{8} \frac{1}{n^4} \]

or:

\[ \frac{8}{16 \cdot 45} L^2 = \sum_{n=1}^{8} \frac{1}{n^4} \]

\[ \frac{L^4}{90} = \sum_{n=1}^{8} \frac{1}{n^4} \quad \text{Riemann Zeta Function} \]
Inhomogeneous boundary conditions (time-independent):

Inhomogeneous boundary conditions/equations, in general:

\[ U_t = \alpha U_{xx} + f(x,t) \]
\[ U(L,t) = \phi_2(t) \]
\[ U(0,t) = \phi_1(t) \]
\[ U(x,0) = g(x) \]

Now, consider a special case where the inhomogeneous B.C.'s do not depend on time, i.e.: \( U(0,t) = U_0 \), \( U(L,t) = U_L \), and there is no sink/source term in the PDE, i.e.: \( f(x,t) = 0 \).
The idea is to solve this problem by decomposing it into a steady-state problem and a homogeneous B.C. problem, i.e.:

\[
U_t = \alpha U_{xx}
\]

\[
W = U_0
\]

\[
V_t = \alpha V_{xx}
\]

\[
V = 0
\]

\[
U(x,0) = g(x)
\]

\[
W(x,0) = U_0
\]

\[
V(x,0) = g(x) - U_0
\]

\[
W_t = 0
\]

\[
V(x,0) = g(x) - U_0
\]

\[
\text{steady-state}
\]

\[
\text{transient}
\]

\[
\text{(particular solution)}
\]

\[
\text{with homog. B.C.}
\]

\[
U(x,t) = W(x) + V(x,t)
\]

Or, in general when \(U(0,t) = \phi_1(t), U(L,t) = \phi_2(t)\)

\[
U_t = \alpha U_{xx}
\]

\[
W_t = \alpha W_{xx}
\]

\[
V_t = \alpha V_{xx} - \frac{(W_0 - \alpha W_{0xx})}{2}
\]

\[
V = 0
\]

\[
W(x,0) = \phi_1
\]

\[
V(x,0) = g(x) - W(x,0)
\]

\[
U(x,t) = W(x,t) + \frac{V(x,t)}{V(x,t)}
\]

a particular solution that satisfies the inhom. B.C.:

\[
W(0,t) = \phi_1(t), W(L,t) = \phi_2(t)
\]

\[
V(x,t)
\]

satisfies the hom. B.C. has the forcing term: \( -W_t + \alpha W_{xx} \)
Example:

PDE: \[ u_t = \alpha u_{xx}, \quad 0 < x < L, \quad t > 0 \]

B.C.: \[ u(0, t) = u_0, \quad u(L, t) = u_1 \]

\[ u_0, u_1: \text{constants} \]

I.C.: \[ u(x, 0) = f(x) \]

Decompose the solution into a steady-state and transient solution:

\[ u(x, t) = w(x) + v(x, t) \]

\[ \uparrow \quad \uparrow \]

\[ \text{satisfies} \quad w(0) = u_0, \quad w(L) = u_1 \]

\[ w_t = 0 \]

\[ \text{satisfies} \quad v_t = \alpha v_{xx} \]

\[ v(0, t) = 0 = v(L, t) \]

\[ \text{Should find an augmented I.C. for this part.} \]

Find the steady-state solution first:

\[ w_t = 0 = \alpha w_{xx} \quad \rightarrow \quad w = Ax + B \]

\[ w(0) = u_0, \quad w(L) = u_1 \]
\[ B = u_0, \quad A L + u_0 = u_1 \Rightarrow A = \frac{u_1 - u_0}{L} \]

\[ \Rightarrow W(x) = \frac{u_1 - u_0}{L} x + u_0 = u_{ss}(x) \]

the steady-state solution.

Now, we have to find \( V(x; t) \), the transient solution:

Let \( u(x; t) = W(x) + V(x; t) \)

PDE: \[ u_t = \frac{W}{t} + v_t = \frac{\lambda^2}{L} (W_{xx} + V_{xx}) \]
\[ \Rightarrow V_t = \lambda^2 V_{xx} \]

B.C.: \[ u(0; t) = u_0 = W(0) + V(0; t) \]
\[ \Rightarrow y_0 = y_0 + V(0; t) \]
\[ \Rightarrow V(0; t) = 0 \]

\[ u(L; t) = u_1 = W(L) + V(L; t) \]
\[ y_1 = y_1 + V(L; t) \]
\[ \Rightarrow V(L; t) = 0 \]

So we have removed the inhomogeneous B.C. from \( V(x; t) \) solution.
I. c. : \( f(x) = u(x,0) = w(x) + v(x,0) \)

\[ \rightarrow \quad v(x,0) = f(x) - w(x) \]

So, we have to solve the following Dirichlet problem to find \( v(x,t) \):

\[ v_t = \frac{\partial^2}{\partial x^2} v_{xx} \quad \text{PDE} \]

\[ v(0,t) = 0 = v(L,t) \quad \text{B. C.} \]

\[ v(x,0) = f(x) - w(x) \quad \text{I. C.} \]

The solution is:

\[ v(x,t) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right) \cdot e^{-\frac{\alpha^2 (n\pi)^2}{L^2} t} \]

\[ b_n = \frac{2}{L} \int_0^L \left\{ f(x) - w(x) \right\} \sin \left( \frac{n\pi x}{L} \right) \, dx \]

And the solution to the inhomogeneous problem is:

\[ u(x,t) = u_0 + \frac{u_1-u_0}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cdot e^{-\frac{\alpha^2 (n\pi)^2}{L^2} t} \]

\[ \text{steady-state} \quad \text{transient} \]
Consider the heat conduction problem with inhomogeneous Neumann B.C.:

PDE: \( U_t = \alpha^2 U_{xx} \)

B.C.: \( U_x(0,t) = q_1, \ U_x(L,t) = q_2 \)

I.C.: \( U(x,0) = f(x) \)

Try to find a steady-state solution:

\( W_t = 0 \rightarrow W_{xx} = 0, \ W = Ax + B \)

\( W'(0) = q_1 = A, \ W'(L) = q_2 \)

So, a steady-state solution exists only if \( q_1 = q_2 \).

Physical interpretation: the system never reaches a steady-state if the heat fluxes at each end are different.
So, the particular solution $w(x,t)$ should also have time dependence.

Guess a simple particular that satisfies the B.C. and the PDE:

$$W(x,t) = Ax^2 + Bx + Ct$$

$$W_x = 2Ax + B, \quad W_{xx} = 2A$$

$$W_t = C$$

We need to have:

$$W_t = C = \alpha^2 W_{xx} = \alpha^2 (2A) \quad \rightarrow \quad C = 2\alpha^2 A$$

$$q_1 = W_x(0,t) = B, \quad q_2 = W_x(L,t) = 2AL + B$$

$$\rightarrow \quad A = \frac{q_2 - q_1}{2L}$$

So, the particular solution is:

$$W(x,t) = \frac{q_2 - q_1}{2L} x^2 + q_1 x + 2\alpha^2 \frac{(q_2 - q_1)}{2L} t$$
Now, let \( u(x,t) = w(x,t) + v(x,t) \)
and find the B.V.P. for \( v(x,t) \):

\[
PDE: \quad u_t = W_t + V_t = \alpha^2 (w_{xx} + v_{xx})
\]

\[\Rightarrow V_t = \alpha^2 v_{xx} \]

\[
B.C.: \quad \begin{align*}
q_1 &= u(x,0) = w_x(0,t) + v_x(0,t) \\
&= \frac{q_2 - q_1}{2} \cdot 2.0 + q_1 + v_x(0,t)
\end{align*}
\]

\[\Rightarrow V_x(0,t) = 0\]

\[
q_2 = u_L(t) = w_x(L,t) + v_x(L,t)
\]

\[= \frac{q_2 - q_1}{2} \cdot 2L + q_1 + v_x(L,t)\]

\[= q_2 + v_x(L,t)\]

\[\Rightarrow V_x(L,t) = 0\]

(The inhomogeneous B.C.'s have been removed from the B.V.P. for \( v(x,t) \))
I. C. : \( p(x) = u(x_{10}) = W(x_{10}) + V(x_{10}) \)

\[ \frac{q_2 - q_1}{2} x + q_1 x + 0 + V(x_{10}) \]

\[ \rightarrow V(x_{10}) = p(x) - \frac{q_2 - q_1}{2L} x^2 - q_1 x \]

\[ = g(x) \quad \text{call it } g(x) \]

The solution to the Neumann B.V.P. for

\( V(x, t) \) is:

\[ V(x, t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \cdot e^{-\frac{2(n\pi)^2}{L^2} t} \]

where:

\[ a_0 = \frac{2}{L} \int_0^L g(x) \cdot dx \]

\[ a_n = \frac{2}{L} \int_0^L g(x) \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot dx \]

And the solution to the problem is:

\[ u(x, t) = W(x, t) + u(x, t) \]

\[ W(x, t) = \frac{q_2 - q_1}{2L} x + q_1 x + 2 \left( \frac{q_2 - q_1}{L} \right) t \]

\[ + \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{2(n\pi)^2}{L^2} t} \]