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A General Limit Theorem for Dynamic Systems with an Application to Population Growth

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ABSTRACT

It is shown in this paper that a certain limiting property of a system of linear homogeneous difference equations is characterized by a solution to a nonlinear system. An application of the result to population growth is also discussed.

1. INTRODUCTION AND SUMMARY

This paper is concerned with the limiting behavior of a solution to a system of linear homogeneous difference equations. It will be shown that under certain conditions $z = \lim_{t\to\infty}(y(t)/\Sigma_{i=1}^{n} y_i(t))$ exists and satisfies a nonlinear system, where $y(t+1) = Ly(t)$, $t = 0, 1, 2, \ldots$, and where z and y are n-column vectors and L is an n by n matrix. A similar result holds for a system of linear homogeneous differential equations. The well-known ergodic theorem of finite Markov chains is a special case of our result.

An application to population growth gives a possible age structure for the so-called zero population growth. Further applications of our result to health services systems are found in Ref. 3.

2. PRINCIPLE RESULT

The following sequence of definitions and theorems is the prerequisite for our main result, Theorem 4. A general treatment of the matrix theory needed in the following is found in Bellman [1] and Lancaster [2].

DEFINITION

A matrix L is non-negative if and only if its elements are all non-negative.

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DEFINITION

An n by n matrix L ($n \geqslant 2$) *is irreducible if and only if its directed graph is strongly connected.*

EXAMPLE

An n by n matrix L ($n \geqslant 2$) given by

$$L = \begin{bmatrix} q_{11} + \beta_1 & \beta_2 & \beta_3 & \dots & \beta_{n-1} & \beta_n \\ q_{12} & q_{22} & 0 & \dots & 0 & 0 \\ 0 & q_{23} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \dots & q_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \dots & q_{n-1,n} & q_{nn} \end{bmatrix}. \quad (1)$$

is non-negative and irreducible if $q_{ii} \geqslant 0$, $q_{i,i+1} > 0$, $q_{ii} + q_{i,i+1} \leqslant 1$ ($1 \leqslant i \leqslant n-1$), $0 \leqslant q_{nn} \leqslant 1$ and $\beta_i > 0$ ($1 \leqslant i \leqslant n$).

DEFINITION

An irreducible non-negative matrix L is primitive if and only if the number of eigenvalues whose moduli are equal to the spectral radius of L is one.

THEOREM 1 (Ref. 2, p. 290)

A non-negative matrix L is primitive if and only if there is a positive integer p such that $L^p > 0$ (*all elements of* L^p *are positive*).

THEOREM 2

The matrix L defined by Eq. (1) *is primitive.*

Proof. By direct calculation we can show that $L^n > 0$. Hence, take $p \equiv n$ in Theorem 1. Q.E.D.

The next theorem is concerned with the existence and uniqueness of a solution to a system of linear and nonlinear equations and inequalities.

THEOREM 3

Let L be an n by n primitive (*hence non-negative*) *and irreducible matrix. Then we have*

$$\begin{bmatrix} \lambda z = Lz & \\ z \geqslant 0, & \sum_{i=1}^{n} z_i = 1 \\ \lambda \geqslant 0 & \end{bmatrix} \Rightarrow \begin{bmatrix} \text{There exist a unique } \lambda \\ \text{and a unique } z \text{ such that} \\ \lambda > 0,\ z > 0 \text{ and } \lambda = \lambda(L) \\ \triangleq \text{spectral radius of } L \end{bmatrix}. \quad (2)$$

where λ *is a scalar and* $z = \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix}$.

The proof of this theorem is found in Nikaido [4] (see Theorem 3.20.1 there). The following theorem explains the limiting behavior of the solution to a system of difference equations.

THEOREM 4

Let L be defined as in Theorem 3, *and let* $y(t)$ *be a solution to*

$$y(t+1) = Ly(t), \qquad (t = 0, 1, 2, \ldots), \tag{3}$$

where

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_n(t) \end{pmatrix},$$

and $y(0)$ ($y(0) \geqslant 0$, $y(0) \neq 0$) *is assumed given. Then a unique*

$$z \triangleq \lim_{t\to\infty} \frac{y(t)}{\sum_{i=1}^{n} y_i(t)}$$

exists, and is given by solving

$$\begin{bmatrix} \lambda z = Lz & \\ z \geqslant 0, & \sum_{i=1}^{n} z_i = 1 \\ \lambda \geqslant 0 & \end{bmatrix}. \tag{4}$$

Proof. We have two cases.

Case 1. L has distinct eigenvalues $\lambda(L), \lambda_2, \lambda_3, \ldots, \lambda_n$, i.e. L is a simple matrix (Ref. 2, p. 61). Then Theorem 2.5.1 of Lancaster (Ref. 2, p. 63) implies that

$$y(t) = L^t y(0) = \left\{\lambda(L)^t A_{\lambda(L)} + \sum_{i=2}^{n} \lambda_i^t A_i\right\} y(0),$$

where

$$A_i = X_i Y_i', \qquad Y_i' X_i = 1, \qquad LX_i = \lambda_i X_i, \qquad Y_i' L = \lambda_i Y_i',$$
$$[i = \lambda(L), 2, 3, \ldots, n] \tag{5}$$

[for simplicity we denote $\lambda_{\lambda(L)}$ by $\lambda(L)$], and where X_i and Y_i are n-dimensional column vectors, A_i is an n by n matrix, and the prime denotes the transpose. It follows that

$$\lim_{t\to\infty} \frac{y(t)}{\sum_{i=1}^{n} y_i(t)} = \lim_{t\to\infty} \frac{\left\{\lambda(L)^t A_{\lambda(L)} + \sum_{i=2}^{n} \lambda_i^t A_i\right\} y(0)}{\sum_{j=1}^{n} \left[\left\{\lambda(L)^t A_{\lambda(1)} + \sum_{i=2}^{n} \lambda_i^t A_i\right\} y(0)\right]_j},$$

$$= \lim_{t\to\infty} \frac{\lambda(L)^t \left\{A_{\lambda(L)} + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda(L)}\right)^t A_i\right\} y(0)}{\lambda(L)^t \sum_{j=1}^{n} \left[\left\{A_{\lambda(L)} + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda(L)}\right)^t A_i\right\} y(0)\right]_j}$$

$$= \frac{A_{\lambda(L)} y(0)}{\sum_{j=1}^{n} [A_{\lambda(L)} y(0)]_j},$$

since $\lambda(L) > |\lambda_i|$ $(2 \leqslant i \leqslant n)$ by the primitivity of L. Further

$$\lambda(L) \frac{A_{\lambda(L)} y(0)}{\sum_{j=1}^{n} [A_{\lambda(L)} y(0)]_j} = \frac{\lambda(L) X_{\lambda(L)} Y'_{\lambda(L)} y(0)}{\sum_{j=1}^{n} [A_{\lambda(L)} y(0)]_j},$$

$$= \frac{L X_{\lambda(L)} Y'_{\lambda(L)} y(0)}{\sum_{j=1}^{n} [A_{\lambda(L)} y(0)]_j} = L \frac{A_{\lambda(L)} y(0)}{\sum_{j=1}^{n} [A_{\lambda(L)} y(0)]_j}.$$

$$\sum_{i=1}^{n} \frac{[A_{\lambda(L)} y(0)]_i}{\sum_{j=1}^{n} [A_{\lambda(L)} y(0)]_j} = 1,$$

and

$$\frac{A_{\lambda(L)} y(0)}{\sum_{j=1}^{n} [A_{\lambda(L)} y(0)]_j} \geqslant 0,$$

since

$$\frac{L^t y(0)}{\sum_{i=1}^{n} [L^t y(0)]_i}$$

is continuous in t. Thus

$$\frac{A_{\lambda(L)} y(0)}{\sum_{j=1}^{n} [A_{\lambda(L)} y(0)]_j}$$

is a non-negative column eigenvector of L corresponding to $\lambda(L)$ with its component sum equal to 1. Theorem 3 implies that

$$z = \frac{A_{\lambda(L)}y(0)}{\sum_{j=1}^{n} [A_{\lambda(L)}y(0)]_j},$$

where z is given by Ref. 4.

Case 2. L has eigenvalues $\lambda(L), \lambda_2, \lambda_3, \ldots, \lambda_s$ $(s < n)$. In this case too $\lambda(L)$ is a simple root of the characteristic equation of L such that $\lambda(L) > |\lambda_i|$ $(2 \leqslant i \leqslant s)$. Thus Theorem 5.4.1 (Sylvester's theorem) of Lancaster (Ref. 2, p. 173) implies

$$L^t = \lambda(L)^t Z_{\lambda(L)} + \sum_{k=2}^{s} \sum_{j=1}^{m_k} \left[\frac{d^{(j-1)}}{d\lambda^{(j-1)}}(\lambda)^t \right]_{\lambda=\lambda_k} Z_{kj}$$

for some matrices $Z_{\lambda(L)}$ and Z_{kj}, where m_k is the multiplicity of eigenvalue λ_k. Since we have

$$\frac{d^{(j-1)}}{d\lambda^{(j-1)}}\lambda^t = t(t-1)\cdots(t-j+2)\lambda^{t-j+1},$$

it follows that

$$L^t = \lambda(L)^t \left[Z_{\lambda(L)} + \sum_{k=2}^{s} \sum_{j=1}^{m_k} t(t-1)\ldots(t-j+2) \left\{ \frac{\lambda_k}{\lambda(L)} \right\}^t \lambda_k^{-j+1} Z_{kj} \right].$$

Since $\lambda(L) > |\lambda_i|$ $(2 \leqslant i \leqslant s)$, the second term in the bracket goes to zero as $t \to \infty$. Thus

$$\lim_{t\to\infty} \frac{y(t)}{\sum_{i=1}^{n} y_i(t)} = \lim_{t\to\infty} \frac{L^t y(0)}{\sum_{i=1}^{n} [L^t y(0)]_i} = \frac{Z_{\lambda(L)}y(0)}{\sum_{i=1}^{n} [Z_{\lambda(L)}y(0)]_i}.$$

If we show $Z_{\lambda(L)} = X_{\lambda(L)} Y'_{\lambda(L)}$, where $X_{\lambda(L)}$ and $Y'_{\lambda(L)}$ are column and row eigenvectors of L corresponding to $\lambda(L)$ respectively, then the rest of the proof reduces to Case 1. By Theorem 5.5.1 (iii) of Lancaster (Ref. 2, p. 177) we have $Z^2_{\lambda(L)} = Z_{\lambda(L)}$; i.e. $Z_{\lambda(L)}$ is an idempotent matrix (Ref. 2, p. 64). Thus Lancaster's Corollary to Theorem 2.11.2 (Ref. 2, p. 83) implies that $Z_{\lambda(L)}$ is necessarily written as $Z_{\lambda(L)} = X_{\lambda(L)} Y'_{\lambda(L)}$ (it is also easy to see that $X_{\lambda(L)}$ and $Y'_{\lambda(L)}$ are column and row eigenvectors of $Z_{\lambda(L)}$, respectively, associated with the eigenvalue one). Q.E.D.

It is noted that Theorem 4 holds regardless of the value of $\lambda(L)$. If $0 < \lambda(L) < 1$, all components of $y(t)$ go to zero as $t \to \infty$, while if $\lambda(L) > 1$, they all diverge to infinity as $t \to \infty$. If $\lambda(L) = 1$, Theorem 4 encompasses the well-known ergodic theorem of the finite Markov chain (Ref. 1, p. 256) (notice that the column sum of L need not be one in our case).

The following two examples illustrate the role played by irreducibility and primitivity in Theorems 3 and 4.

EXAMPLE

Let $y(t)$ be given by

$$y(t+1) = Ly(t), \qquad (t = 0, 1, 2, \ldots),$$

where

$$y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 0.9 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $y_2(t) = 1$, $(t = 0, 1, 2, \ldots)$, and $y_1(t)$ strictly increases as $t \to \infty$. Hence

$$z_1 = \lim_{t \to \infty} \frac{y_1(t)}{y_1(t) + y_2(t)} = 1,$$

and

$$z_2 = \lim_{t \to \infty} \frac{y_2(t)}{y_1(t) + y_2(t)} = 0$$

although $y_2(t) > 0$ for all t. Note that L is not irreducible. Thus, Theorem A3 does not apply (hence, $z \not> 0$).

EXAMPLE

Let $y(t)$ be given by

$$y(t+1) = Ly(t), \qquad (t = 0, 1, 2, \ldots),$$

where

$$y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$y(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad y(2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad y(3) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ldots.$$

Hence there is no z such that

$$z = \lim_{t \to \infty} \frac{y(t)}{y_1(t) + y_2(t)}.$$

We note that L is not primitive; i.e. there is no $n < \infty$. Therefore Theorem 4 does not apply.

3. AN APPLICATION TO POPULATION GROWTH [1]

As was pointed out in Ref. 3, the λ given by Theorem 4 corresponds to population growth at the rate $\lambda - 1$. In the following, we shall derive an explicit expression of z given by Eq. (4) for the case where $\lambda(L) = 1$. This case might be particularly interesting in accordance with movement toward the so-called zero population growth rate, since we can derive the exact age structure of the population in equilibrium as a function of β_i, q_{ij} and $y_i(0)$ $(1 \leqslant i, j \leqslant n)$. Denoting by z_i and t_i the ith components of column and row eigenvectors of L corresponding to $\lambda(L) = 1$ respectively, we have from Eq. (4) that:

$$
\begin{aligned}
(1 - q_{22})z_2 &= q_{12}z_1, \\
(1 - q_{33})z_3 &= q_{24}z_2, \\
&\;\;\vdots \\
(1 - q_{nn})z_n &= q_{n-1,n}z_{n-1}, \\
q_{12}y_2 &= (1 - q_{11} - \beta_1)y_1, \\
q_{23}y_3 &= (1 - q_{22})Y_2 - \beta_2 y_1, \\
&\;\;\vdots \\
q_{n-1,n}y_n &= (1 - q_{n-1,n-1})y_{n-1} - \beta_{n-1}y_1.
\end{aligned} \tag{6}
$$

The first $n - 1$ equations of Eq. (6) and $\Sigma_{i=1}^{n} z_i = 1$ can be solved recursively to give

$$
\begin{aligned}
z_1 &= [1 + \textstyle\sum_{j=2}^{n} \pi_{i=1}^{j-1} q_{i,i+1}/(1 - q_{i+1,i+1})]^{-1}, \\
z_j &= z_1 \pi_{i=1}^{j-1} q_{i,i+1}/(1 - q_{i+1,i+1}), \qquad (2 \leqslant j \leqslant n),
\end{aligned} \tag{7}
$$

and the last $n - 1$ equations of Eq. (6) and $\Sigma_{i=1}^{n} z_i t_i = 1$ give

$$
\begin{aligned}
t_1 &= [z_1\{1 + \textstyle\sum_{j=2}^{n} C_j \pi_{i=1}^{j-1} q_{i,i+1}/(1 - q_{i+1,i+1})]\}^{-1}, \\
t_j &= C_j t_1, \qquad (2 \leqslant j \leqslant n),
\end{aligned}
$$

where

$$
\begin{aligned}
C_2 &= (1 - q_{11} - \beta_1)/q_{12}, \\
C_3 &= [-\beta_2/q_{23} + (1 - q_{11} - \beta_1)(1 - q_{22})/q_{12}q_{23}], \\
C_j &= [-\beta_{j-1}/q_{j-1,j} - \textstyle\sum_{k=2}^{j-2}\beta_k \pi_{i=k+1}^{j-1}(1 - q_{ii})/\pi_{i=k}^{j-1} q_{i,i+1} \\
&\quad + (1 - q_{11} - \beta_1)\pi_{i=2}^{j-1}(1 - q_{jj})/\pi_{i=1}^{j-1} q_{i,i+1}], \qquad (4 \leqslant j \leqslant n).
\end{aligned}
$$

[1] In this section L is assumed to be given by Eq. (1) in which β_i is the birth rate of the people of the ith age group, q_{ii} the probability that the people of the ith age group remain within the group, $q_{i,i+1}$ the probability that they move to the next age group within a unit time period. $y_i(t)$ is the number of people in the ith age group at time t. z_i's give a stable age distribution of the total population. Also the death rate is given by $1 - (q_{ii} + q_{i,i+1})$ for $i = 1, 2, \ldots, n - 1$, and $1 - q_{nn}$.

We have by Eq. (5) that

$$y(\infty) = \lim_{t\to\infty} y(t) = A_{\lambda(L)}y(0) = \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix} (t_1, t_2, \ldots, t_n)y(0) \tag{8}$$

$$= \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix} \sum_{i=1}^{n} t_i y_i(0).$$

An interpretation of Eq. (8) is that given the present age structure $y(0)$ of the population, $y(t)$ will approach $y(\infty)$ if $\lambda(L) = 1$, where $\Sigma_{i=1}^{n} t_i y_i(0)$ represents the number of the total population at the equilibrium. Finally, we note that Eq. (4) implies [2] that $\lambda(L) = 1$ if $q_{ii} + q_{i,i+1} + \beta_i - 1 = 0$ $(1 \leqslant i \leqslant n)$ and $q_{nn} + \beta_n - 1 = 0$.

4. A GENERAL LIMIT THEOREM FOR LINEAR DIFFERENTIAL EQUATIONS

Consider a system of linear differential equations

$$\begin{aligned} &x(t) = Ax(t), \qquad (0 \leqslant t < \infty), \\ &x(0) \text{ given}, \end{aligned} \tag{9}$$

where A is an n by n constant matrix and

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix}.$$

Let $z(t)$ be defined by

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \cdot \\ \cdot \\ \cdot \\ z_n(t) \end{pmatrix} = \frac{x(t)}{\sum_{j=1}^{n} x_j(t)}.$$

Then the following theorem holds.

[2] It is shown [3] that $\lambda - 1 = (q_{11} + q_{12} + \beta_1 - 1)z_1 + (q_{22} + q_{23} + \beta_2 - 1)z_2 + \ldots + (q_{n-1,n-1} + q_{n,n} + \beta_{n-1} - 1) + (q_{nn} + \beta_n - 1)z_n$.

THEOREM 5

Suppose that A has a real eigenvalue λ_1 with simple multiplicity such that

$$\max_{2 \leq i \leq k} [\mathrm{Re}\{\lambda_i\}, \quad \mathrm{Im}\{\lambda_i\}] < \lambda_1, \tag{10}$$

where k is the number of distinct eigenvalues of A. Then there exists a unique $z = \lim_{t\to\infty} z(t)$ which is independent of $x(0)$. Further z is the eigenvector of A corresponding to λ_1.

Proof. It is known by the theory of ordinary differential equations (Ref. 5, pp. 94–99) that $x(t)$, the solution to Eq. (9), can be written for some constants $c_1, c_2, \ldots, c_n$ as follows:

$$x(t) = c_1 x^1(t) + c_2 x^2(t) + \cdots + c_n x^n(t),$$

where

$$x^{k_1} = h_{k_1} e^{\lambda_1 t}, \qquad (k_1 \equiv 1),$$

$$x^{k_1+1} = h_{k_1} e^{\lambda_2 t},$$

$$\vdots$$

$$x^{k_1+k_2} = \left(\frac{t^{k_2}}{k_2!} h_{k_1+1} + h_{k_1+k_2}\right) e^{\lambda_2 t},$$

$$\vdots$$

and where h_{k_1} is a basis set with the eigenvalue λ_1, $h_{k_1+1}, h_{k_1+2}, \ldots, h_{k_1+k_2}$ are a basis set with the eigenvalue $\lambda_2, \ldots$, and $k_1 \equiv 1, k_2, \ldots$ are the multiplicities of $\lambda_1, \lambda_2, \ldots$, respectively. It follows that

$$z(t) = \frac{\sum_{i=1}^{n} c_i x^i(t)}{\sum_{j=1}^{n} \sum_{i=1}^{n} c_i x_j^i(t)} = \frac{e^{\lambda_1 t} \sum_{i=1}^{n} c_i x^i(t) e^{\lambda_1 (t)}}{e^{\lambda_1 t} \sum_{j=1}^{n} \sum_{i=1}^{n} c_i x_j^i(t) e^{\lambda_1 (t)}},$$

$$= \frac{\sum_{i=1}^{n} c_i x^i(t) e^{\lambda_1 (t)}}{\sum_{j=1}^{n} \sum_{i=1}^{n} c_i x_j^i(t) e^{\lambda_1 (t)}}.$$

Since each $x^i(t) e^{\lambda_1 (t)}$ is of the form $f_i(t) e^{(\lambda_i - \lambda_1)t}$ with $f_i(t)$, a vector function of the nth order, being at most of order t^{k^*} where

$$k^* = \max(k_1, k_2, \ldots, k_k) \leq n.$$

From Eq. (10) it follows that

$$z = \lim_{t \to \infty} z(t) = \frac{h_1}{\sum_{i=1}^{n} h_{1i}}$$

is unique. Q.E.D.

Thus it is seen that under the assumptions of Theorem 5 z is the solution to a nonlinear programming problem

$$\max \lambda$$

$$\text{subject to } Az = \lambda z \quad \text{and} \quad \sum_{i=1}^{n} z_i = 1. \tag{11}$$

The nonlinear program (11) is equivalent to the well-known power method used to derive the maximum eigenvalue and the corresponding eigenvector of A. The power method starts with an arbitrary x^0. Next x^p for $p = 1, 2, \ldots$ is defined by $x^1 = Ax^0$, $x^2 = Ax^1, \ldots, x^{p+1} = Ax^p, \ldots$. λ and z satisfying (11) are approximately given by

$$\lambda = \frac{(x^p)^T(x^{p+1})}{(x^p)^T(x^p)} \quad \text{and} \quad z = \frac{x^p}{\sum_{i=1}^{n} x_i^p},$$

where p is assumed to be sufficiently large and T denotes a transpose.

We can state a somewhat stronger theorem if A is a certain non-negative matrix.

THEOREM 6

Let A be primitive and irreducible. Then Theorem 5 holds. Further, if $x(0) \geqslant 0$ and $x(0) \neq 0$, then

$$z = \lim_{t \to \infty} \frac{x(t)}{\sum_{j=1}^{n} x_j(t)} > 0.$$

Proof. The irreducibility of A implies that A has a λ_1 satisfying the conditions of Theorem 5. Hence Theorem 5 holds. For a given $x(0)$, we have

$$x(t) = e^{At}x(0),$$

where

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots$$

is a matrix exponential. Since A is primitive, there exists a finite integer $l > 0$ such that $A^l > 0$. Hence $e^{At} > 0$ $(0 < t < \infty)$; that is, all elements of e^{At} $(t > 0)$ are positive and strictly increasing as $t \to \infty$. Given $x(0)$ such that $x(0) \geqslant 0$ and $x(0) \neq 0$, we have $x_i(t) > 0$ and $\lim_{t\to\infty} x_i(t) = \infty$ $(1 \leqslant i \leqslant n)$. It follows by the continuity that

$$\lim_{t\to\infty} \frac{x(t)}{\sum_{i=1}^{n} x_i(t)} = z > 0. \qquad \text{Q.E.D.}$$

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REFERENCES

1 R. E. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York (1960).
2 P. Lancaster, *Theory of Matrices*, Academic, New York (1969).
3 M. Nakamura, Mathematical Analysis and Optimization of Health Services Systems, Ph.D. Dissertation, Johns Hopkins University, Baltimore Md., 1972.
4 H. Nikaido, *Introduction to Sets and Mapping in Modern Economics*, North Holland-American Elsevier, Amsterdam and New York (1970).
5 L. S. Pontryagin, *Ordinary Differential Equations*, Addison-Wesley, Reading, Mass. (1962).