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ON A CLASS OF STOCHASTIC OPTIMIZATION PROBLEMS WITH A SPECIFIED GROWTH PATTERN*

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We consider a system which consists of several subsystems. The outputs of these subsystems satisfy linear difference equations which specify the growth pattern of the output of the system over time. The state of each subsystem is described by a finite Markov chain, the transition probabilities of which are subject to our control. Associated with the Markov chain of each subsystem is a cost per unit output of the subsystem, and the cost is incurred as the subsystem occupies one of J states in each epoch. The problem of minimizing the total expected cost with respect to the transition probabilities over a sufficiently long period of time is shown under certain conditions to reduce to a collection of n independent programs. Each of these can be solved by column generation techniques.

1. Introduction

The system of interest to us is a collection of n subsystems. The evolution of each subsystem is probabilistic and forms a finite Markov chain. Associated with each subsystem is an output level which, together with the output levels of other subsystems, satisfies a system of linear difference equations describing the interrelationships among the subsystems. An optimization problem based on this model is formulated and the conditions under which this problem reduces to n independent programs are given.

2. The Model

Let us consider a system which consists of n subsystems. Each subsystem can occupy one of J states that are numbered 1 through J. The evolution of each subsystem is probabilistic and independent of the others. Let

$$(1) \qquad \pi_j^{(i)}(t) = \text{Prob. [subsystem } i \text{ occupies state } j \text{ in epoch } t]$$

for $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, J$ and $t = 0, 1, 2, \cdots$. The evolution of subsystem i from each epoch to the next is governed by a stochastic matrix $P^{(i)}$, and

$$(2) \qquad \pi^{(i)}(t+1) = \pi^{(i)}(t)P^{(i)} \quad \text{for all} \quad i, t,$$

where $\pi^{(i)}(t) = (\pi_1^{(i)}(t), \cdots, \pi_J^{(i)}(t))$, and where $\pi^{(i)}(0)$ such that $\pi_j^{(i)}(0) \geqq 0$ and $\sum_{j=1}^{J} \pi_j^{(i)}(0) = 1$ are assumed given for $i = 1, 2, \cdots, n$. The decision variables in this model are the matrices $P^{(i)}$, and we assume that

$$(3) \qquad P^{(i)} \in S^{(i)} \quad \text{for all} \quad i,$$

where $S^{(i)}$ is a convex polyhedron.

The manner in which the subsystems are interrelated will now be described. Let L be a fixed nonnegative primitive and irreducible $n \times n$ matrix.[1] Let $y(0)$ be a fixed

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[1] A nonegative matrix L is irreducible if and only if its directed graph is strongly connected. An irreducible matrix L is primitive if and only if there exists a positive integer p ($< \infty$) such that $L^p > 0$. See Gantmacher [5] or Lancaster [6] for a general treatment of nonnegative matrices.

nonnegative, nonzero $n \times 1$ vector. Let $y(t)$ be defined recursively by

$$y(t) = Ly(t-1) \tag{4}$$

for $t = 1, 2, \cdots$. Let the ith component of $y(t)$ be interpreted as the output level of subsystem i in epoch t. The number c_j^i is the cost per unit output of subsystem i when it occupies state j. So, with $c^i = (c_1^i, \cdots, c_J^i)'$ where the prime denotes the transpose, the quantity

$$y_i(t)\pi^{(i)}(t)c^i \tag{5}$$

is the expected cost of subsystem i in epoch t. (We can also consider the cost associated with the transition of each subsystem from one state to another, but this adds nothing to the model itself.)

Define as a (stationary) policy P an n-tuple of stochastic matrices: that is, $P = (P^{(1)}, \cdots, P^{(n)})$ with $P^{(i)} \in S^{(i)}$ for each i. Let α be the discount factor. Then the expected discounted cost $J(T, P)$ of using policy P in epochs 0 through T is given by

$$J(T, P) = \sum_{t=0}^{T} \alpha^t \sum_{i=1}^{n} y_i(t)\pi^{(i)}(t)c^i = \sum_{t=0}^{T} \alpha^t \sum_{i=1}^{n} y_i(t)\pi^{(i)}(0)[P^{(i)}]^t c^i. \tag{6}$$

Policy P^* is said to overtake policy P if there exists an integer T^* such that, for all $T > T^*$,

$$J(T, P) > J(T, P^*) \tag{7}$$

(cf. Gale [4]).

Before giving conditions under which one policy overtakes all others, we will consider the following examples. Consider an n-sector economy. $y_i(t)$ could be the output in epoch t with the technological level of the ith sector equal to one of J levels $\{1, 2, \cdots, J\}$. The growth pattern of the output $y(t)$ of the economy could be characterized by (4) with the given matrix L, and the technological transitions within each sector, classified into J levels $\{1, 2, \cdots, J\}$, given by (2). c_j^i would be the cost per unit output of the ith sector incurred when the ith sector of the economy finds itself in technological level j. Assuming that the growth pattern of the economy as a whole is specified by (4), our problem would be to find an optimal n-tuple of stochastic matrices P for the technological level such that the total expected cost (6) due to technological change is minimized subject to (2) and (3).

As another example, consider a community health services system in which $y_i(t)$ is the number of people in the community in age group i in epoch t. The state of health of a representative individual of age group i in each epoch is equal to one of J values $\{1, 2, \cdots, J\}$ which correspond to the possible health states of an individual. The matrix L characterizes the growth pattern of the population, and the (l, k)th element of $P^{(i)}$ is the transition probability that an individual of age group i will move to health state k given that he was in health state l in the previous epoch. A health services cost c_j^i is incurred by the community when an individual of age group i is in health state j. Assuming that $P^{(i)}$'s are to some extent controllable by public community health policies, our problem is to minimize the total expected health services cost (6) subject to (2), (3) and the population growth pattern (4). (For an example of a Markovian description of a community health services system see Navarro, Parker and White [9].)

One might note that our model is a combination of a deterministic system (4) and the finite Markov chains of its subsystems which are characterized by (2). It is easy to see that if there is only one state (i.e. $J = 1$) for each subsystem, the dynamics of the system are fully described by (4), while if there is only one subsystem (i.e. $n = 1$) our model reduces to a class of Markovian decision problems (see Wolfe and Dantzig [10], Denardo [1] and Denardo and Fox [2]).

In the following section conditions are given under which one policy overtakes all others.

3. Our Main Theorem

Let us first analyze the matrix L which is nonnegative, primitive and irreducible. It is well known (Gantmacher [5] and Lancaster [6]) that the eigenvalue, λ, whose magnitude is largest, is positive and unique, and that this eigenvalue corresponds to a single right eigenvector z, which is strictly positive. Furthermore, for any $n \times 1$ nonzero vector y,

$$\lim_{t\to\infty} (L^t y/\lambda^t) = kz, \tag{8}$$

where k is a nonzero constant.

We now impose the condition that

$$\lambda\alpha > 1. \tag{9}$$

This means that the asymptotic growth rate of the system more than offsets the discount factor.[2] Consider the following n programs:

$$\text{Minimize} \quad \pi^{(i)} c^i, \tag{10}$$

subject to the constraints

$$\pi^{(i)} = \pi^{(i)} P^{(i)}, \tag{11}$$

$$P^{(i)} \in S^{(i)}, \tag{12}$$

$$\pi^{(i)} \geqq 0, \qquad \textstyle\sum_{j=1}^{J} \pi_j^{(i)} = 1. \tag{13}$$

We assume that all n programs have the following properties: the ith program is maximized by a unique matrix $P^{*(i)}$, and $S^{(i)}$ consists exclusively of irreducible, primitive matrices.[3]

THEOREM. *Policy $P^* = (P^{*(1)}, \cdots, P^{*(n)})$ overtakes all others.*

PROOF. Consider policy $P \neq P^*$. We have from (2) that

$$\pi^{(i)}(t) = \pi^{(i)}(0)[P^{(i)}]^t \tag{14}$$

and

$$\pi^{*(i)}(t) = \pi^{(i)}(0)[P^{*(i)}]^t. \tag{15}$$

It is well known (see, for example, Feller [3, p. 389]) that the limits of $\pi^{(i)}(t)$ and $\pi^{*(i)}(t)$ as $t \to \infty$ exist under our assumptions and are characterized by the relationships (11)—(13) for P and P^*, respectively: that is,

$$\pi^{(i)} = \lim_{t\to\infty} \pi^{(i)}(t) \tag{16}$$

[2] As for interpreting λ as the growth factor see, for example, Nakamura [8].

[3] An irreducible, primitive stochastic matrix is called "acyclic" or "aperiodic" (see, for example, Feller [3]). In certain cases it is possible to verify the aperiodicity and irreducibility of $P^{(i)}$ before we solve the problem. One such case, for instance, occurs when all elements of $P^{(i)}$ are forced to be positive.

and

(17) $$\pi^{*(i)} = \lim_{t\to\infty} \pi^{*(i)}(t).$$

Since the programs have unique solutions, there must exist a T and a positive ϵ such that

(18) $$\textstyle\sum_{i=1}^{n} z_i \pi^{(i)}(t) c^i > \sum_{i=1}^{n} z_i \pi^{*(i)}(t) c^i + \epsilon$$

for all $t > T$. Since $y(0)$ is positive, the scalar in (8) must be positive. Moreover, since

(19) $$y_i(t)/\lambda^t \cong k z_i\,,$$

it must follow from (6), (9) and (18) that policy P^* performs better in epoch t than policy P by an amount of roughly $k(\lambda\alpha)^t\epsilon$, which diverges with t to plus infinity. This completes the proof of the theorem.

4. Concluding Remarks

We have studied a class of programming problems which might occur in a system consisting of n subsystems. The outputs of the subsystems are interrelated by linear difference equations which specify the growth pattern of the output of the system as a whole. The state of each subsystem is described by a finite Markov chain which incurs a cost for each unit output of the subsystem as it occupies one of J states. The problem of minimizing the total expected cost with respect to an n-tuple of stochastic matrices over a sufficiently long period of time has been shown under certain conditions to reduce to a collection of n independent programs. Each of these can be solved by column generation techniques, and with that interpretation they become linear programs to which a policy-improvement-type technique is applied (see, for example, Mine and Osaki [7]). Finally we note that the basic idea in this paper can be applied to more general models of the type considered by Denardo [1], Denardo and Fox [2] and Wolfe and Dantzig [10].

References

1. Denardo, E. V., "On Linear Programming in a Markov Decision Problem," *Management Science*, Vol. 16, 1970, pp. 165–177.
2. —— and Fox, B. L., "Multichain Markov Renewal Programs," SIAM *Journal of Applied Mathematics*, Vol. 16, 1968, pp. 468–487.
3. Feller, W., *An Introduction to Probability Theory and Its Applications*, Vol. 1 (Third edition), John F. Wiley and Sons, New York, 1968.
4. Gale, D., "On Optimal Development in a Multi-sector Economy," Review of Economic Studies, Vol. 31, 1964, pp. 1–24.
5. Gantmacher, F. R., *The Theory of Matrices*, Vol. 2, Chelsea, New York, 1960.
6. Lancaster, P., *Theory of Matrices*, Academic Press, New York, 1969.
7. Mine, H. and Osaki, S., *Marovian Decision Processes*, American Elsevier, New York, 1970.
8. Nakamura, M., "A General Limit Theorem for Dynamic Systems with an Application to Population Growth," *Mathematical Biosciences*, Vol. 16, 1973, pp. 177–187.
9. Navarro, V., Parker, R. and White, K., "A Stochastic and Deterministic Model of Medical Care Utilization," *Health Services Research*, Vol. 5, 1970, pp. 342–357.
10. Wolfe, P. and Dantzig, G., "Linear Programming in a Markov Chain," *Operations Research*, Vol. 10, 1962, pp. 702–710.