Applications of the Fundamental Group to Cell Complexes

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Abstract

In this work, we discuss about some applications of the notion of fundamental group to cell complexes. We prove that the fundamental group of a space X does not change after we have attached n-cells to it with $n > 2$. When $n = 2$, it does and we have an explicit relationship between the fundamental groups involved. As a consequence of that result, we show that the fundamental group of a cell complex X is reduced to that of its 2-skeleton X^2, that is, $\pi_1(X, x_0) \cong \pi_1(X^2, x_0)$. As application, we show that for two given closed and path-connected surfaces S_g and S_h both either orientable or nonorientable and of genus g and h respectively, a necessary condition for them to be homeomorphic is that their genera must be equal, that is, $g = h$. Furthermore, we prove that considering the presentation $\langle g_\alpha | r_\beta \rangle$ of a group G, there exists a space X_G such that $\pi_1(X_G) \cong G$. That space is a 2-dimensional cell complex whose 1-skeleton is the wedge sum $X^1 = \vee_\alpha S_\alpha^1$ as many copies of the unit circle S^1 as generators g_α's, and we complete the construction of X_G by attaching 2-cells e_β^2 to X^1 along the loops associated to the relators r_β. We end the work with a description of the space X_G when $G = \langle a | a^n \rangle, n \geq 1$. We prove that X_G is homeomorphic to a simple space, and we then notice that it is a surface only when $n = 1, 2$.

Key words: Fundamental group, cell complex, closed and path-connected surface.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Arnaud NGOPNANG NGOMPE, 18 May 2018.

Résumé

Dans le présent travail, nous traitons de quelques applications de la notion de groupe fondamental aux complexes cellulaires comme le stipule le titre. Nous prouvons que le groupe fondamental d'un espace X ne change pas après qu'on y ai attaché des n-cellules avec $n > 2$. Lorsque $n = 2$, cela change et l'on a une relation explicite entre ces groupes fondamentaux. Comme conséquence de ce résultat, nous démontrons que le groupe fondamental d'un complexe cellulaire X se réduit à celui de son 2-squelette X^2, c'est à dire, $\pi_1(X, x_0) \cong \pi_1(X^2, x_0)$. Comme application, nous démontrons qu'étant donné deux surfaces S_g et S_h fermées et connexes par arc, toutes deux orientables où nonorientable et de genres g et h respectivement, une condition nécessaire pour qu'elles (surfaces S_g and S_h) soient homéomorphe est que leurs genres soient égaux, c'est à dire, $g = h$. Nous prouvons aussi qu'en considérant la représentation $\langle g_\alpha | r_\beta \rangle$ du groupe G, il existe un espace X_G tel que $\pi_1(X_G, x_0) \cong G, x_0 \in X_G$ est un complexe cellulaire dimension 2 dont le 1-squelette est la "wedge sum" (somme de coins en Français) $X^1 = \vee_\alpha S^1_\alpha$ d'autant de copies du cercle unité S^1 que de générateurs g_α's, et l'on complète la construction de X_G en attachant des 2-cellules e^2_β à X^1 le long des chemins fermés associés aux relations (ou relateurs) r_β. Nous clôturons ce travail par un exemple de description de l'espace X_G avec $G = \langle a | a^n \rangle, n \geq 1$; Ainsi X_G est homéomorphe à un simple espace, et nous remarquons que c'est une surface uniquement quand $n = 1, 2$.

Mots Clés: Groupe fondamental, complexe cellulaire, surface fermée et connexe par arc.

Contents

1. Introduction

Algebraic topology can be defined as a branch of Mathematics which uses tools from abstract algebra to study topological spaces. The basic goal of algebraic topology is to find algebraic invariants which classify topological spaces up to homeomorphism, though usually most classifications are only up to homology equivalence. Among the algebraic invariants studied in Algebraic topology are fundamental group, homology group and cohomology group. In this thesis we are concerned only with the fundamental group.

The fundamental group is a group associated to any pointed, path-connected topological space. It was introduced by Henri Poincaré [9] in his attempt to classify topological spaces. For examples, it is a well known result that the fundamental group is a complete invariant for compact topological spaces of dimension 2. The famous Poincaré conjecture states that "Every closed and simply-connected 3-manifold is homeomorphic to the 3-sphere" . This conjecture was proved in 2002 by Gregory Perelman [14], an achievement that earned him a fields medal in 2014.

1.1 Objectives

In this thesis, we will study fundamental group in relation to cell complexes. A cell complex can be defined roughly as a type of topological space made of basic building blocks called cells. Fundamental group will help us to prove some results and to answer a natural question associated to cell complexes. Hence, we will alternatively, prove two results which give a necessary condition for two closed and connected surfaces (orientable and nonorientable) to be homeomorphic. Moreover, we will answer to the following natural question: for a given group G, can we find a topological space X_G such that its fundamental group and group G are isomorphic? We will see that the answer to this question is "yes", and the space X_G is a cell complex by construction. The rest of the work is subdivided into four chapters.

1.2 Outline of Thesis

In Chapter 2, we will recall some basic concepts of group theory and also the concept of fundamental group which will allow us to make constructions and proofs thereafter. In Chapter 3, a description of the notion of cell complex will be given, followed by the proof of a more general result, which will allow us later to compute fundamental groups of some cell complexes. Chapter 4 consists in the presentation of some applications of fundamental group of cell complexes. So we will make a brief discussion about classification of surfaces which are cell complexes and we will finish the chapter by finding and describing the space X_G. In the last chapter, Chapter 5, we will provide a general conclusion to this work.

2. Preliminaries

To achieve the goals we have set so far, we first need to describe the tools needed. So in this chapter, we present alternatively notions of free product of groups and group presentation which are from group theory. Also, we give a brief definition of the concept of fundamental group followed by the examples of computing the one of a wedge sum of finitely many copies of the unit circle S^1 and the one of the n-sphere ($n \geq 2$).

2.1 Some Group Theory Basics

The group presentation is very useful in the last part of Chapter 4 where for the construction of the space X_G, we need to have G in the form of group presentation, that is, $G = \langle g_\alpha | r_\beta \rangle$, where g_α's are generators and r_β's are relators (See Definition 2.1.5).

We begin with the statement of the first isomorphism theorem which will help us to prove some results below.

2.1.1 Theorem (First Isomorphic Theorem). *[6] If $\theta : G \to H$ is a group homomorphism, then $G/ker\theta \cong im\theta$.*

2.1.2 Definition (Generated Subgroup). [6] Let $X \subseteq G$, with G a group. Consider the set $X^{-1} = \{x^{-1} \mid x \in X\}$ and set $A = X \cup X^{-1}$. Define A^* to be the set of all words over A. Elements of A^* represent elements of G and it is closed under concatenation and inversion. So it is a subgroup of G called subgroup of G generated by X and denoted $A^* = \langle X \rangle$, where the empty word represents the neutral element 1_G of G.

2.1.3 Definition (Free Group on a Set). [6] Let F be a group and $X \subseteq F$. Then F is said to be free on X if for any group G and map $\theta : X \to G$, there exists a unique homomorphism $\theta' : F \to G$ with $\theta'(x) = \theta(x)$, for all $x \in X$.

2.1.4 Proposition. [6] If F is a free group on X, then $F = \langle X \rangle$ (that is, X generates F).

2.1.5 Definition (Presentation of a Group). [6] Let G be a group. A presentation of G is an isomorphism $G \cong \langle g_\alpha \mid r_\beta \rangle$ of G with a free group generated by g_α modulo the normal subgroup $\langle\langle r_\beta \rangle\rangle$ generated by a finite number of words r_β formed from $\{g_\alpha\}$.

g_α's and r_β's are respectively called generators and relators.

2.1.6 Remark. [6] We need to notice here that concatenation in the free group $\langle g_\alpha | \rangle$ descends to the group operation on G in the quotient. We think of the presentation as giving a way to multiply elements of G freely, subject to relations $r_\beta = 1$.

2.1.7 Proposition. [6] All groups have presentations and finite groups have finite presentations.

Proof. (**Proposition 2.1.7**). Let G be a group. Choose a generator subset $X \subset G$, that is $G = \langle X \rangle$. Let F be free group of X. By definition of the free group, a map $\theta : X \to G$ such that $\theta(x) = x$, for all $x \in X$ extends to a surjective map $\theta' : F \to G$, then from the first isomorphic theorem, we have $G \cong F/N$ where $N = ker\theta$. Let us now choose a generator subset $R \subset N$, that is, $N = \langle\langle R \rangle\rangle$ (This double angle notation is due to the fact that N is normal). Therefore $G = \langle X \mid R \rangle$.

Assume that G is finite. In this particular case, a finite generator subset $X \subset G$ is chosen and then $|F \cdot N| = |G|$ is finite hence a finite generator subset $R \subset N$ since N is finite in G. $\square$

2.1.8 Definition (Commutator). [6] Let G be a group and let $x, y \in G$. The commutator of x and y is $xyx^{-1}y^{-1}$ and it is denoted $[x, y]$.

2.1.9 Definition (Commutator Subgroup). [8] The commutator subgroup of G is the normal subgroup denoted $[G, G]$ generated by the commutators $[a, b] = aba^{-1}b^{-1}$, $a, b \in G$, that is,

$$[G, G] = \langle\langle\{[a, b] \mid a, b \in G\}\rangle\rangle. \tag{2.1.1}$$

2.1.10 Definition (Abelianization of a Group). [8] Let G be a group. The abelianization of G is the quotient group $Ab(G)$ given by:

$$Ab(G) = G/[G, G]. \tag{2.1.2}$$

2.1.11 Example. Consider the group presentation given by $G = \langle a, b | a^n, b^m \rangle$. The abelianization of the group G is given by

$$Ab(G) = \langle a, b | a^n, b^m \rangle / \langle\langle[a, b]\rangle\rangle \cong \langle a, b | a^n, b^m, [a, b] \rangle.$$

The notion of free group is relevant when we compute the fundamental group of some spaces. It is mainly used by the van Kampen theorem which is one of the main tools used to compute fundamental groups. Before defining the free product of groups, we first define the notion of disjoint union.

2.1.12 Definition (Disjoint Union). Let A and B be sets. The disjoint union $A \amalg B$ of A and B is defined as follows:

- If $A \cap B = \emptyset$, then $A \amalg B = A \cup B$.

- If $A \cap B \neq \emptyset$, we first take isomorphic copies A' and B' of A and B such that $A' \cap B' = \emptyset$ ($A' = A \times \{0\}$ and $B' = B \times \{1\}$ for instance) and then $A \amalg B = A' \cup B'$.

2.1.13 Definition (Word and its Length). Let $(G_1, \bullet)$ and $(G_2, \star)$ be groups with $e_1 \in G_1$ and $e_2 \in G_2$ as identity elements.

- A word over the alphabet $G_1 \amalg G_2$ is a sequence $w = a_1 a_2 \cdots a_k$, $k \geq 0$ and $a_i \in G_1 \amalg G_2$, $\forall i = 1, 2, \cdots, k$.

- k is the length of the word.

- A word w is reduced if the following are verified:

  - Adjacent letters are from different groups, that is, $a_i \in G_1$ implies $a_{i-1} \in G_2$ and $a_i \in G_2$ implies $a_{i-1} \in G_1$.
  - Identity elements do not appear, that is, $a_i \neq e_1$ and $a_i \neq e_2$.

Reduction algorithm

From any word $w = a_1 \cdots a_{i-1} a_i a_{i+1} a_{i+2} \cdots a_k$, it is possible to obtain a unique reduced word w' by applying the following two steps:

- If for any $i = 1, 2, \cdots, k, a_i, a_{i+1} \in G_1$ (respectively $a_i, a_{i+1} \in G_2$), then combine a_i and a_{i+1} in G_1 (respectively in G_2), that is, $b = a_i \bullet a_{i+1}$ (respectively $b = a_i \star a_{i+1}$). We get a word of smaller length $w = a_1 \cdots a_{i-1} b a_{i+2} \cdots a_k$.

- If any of the letters a_i of the word w is the identity e_1 or e_2, we simply take it out.

2.1.14 Definition (Free Product)**.** Let G_1 and G_2 be groups.

- The free product of G_1 and G_2 is $G_1 * G_2 = \{w \text{ word} \mid w \text{ is a reduced word over } G_1 \amalg G_2\}$.

- Let e denote the empty word (that is, the word of length $k = 0$).

- Let us consider the binary operation $\diamond$ defined by:

$$\diamond : G_1 * G_2 \times G_1 * G_2 \to G_1 * G_2$$
$$(a_1 \cdots a_k, b_1 \cdots b_s) \mapsto c, \tag{2.1.3}$$

where $c =$ reduction of $(a_1 \cdots a_k b_1 \cdots b_s)$.

The binary operator $\diamond$ is well-defined since the reduced word is unique for a given word.

2.1.15 Proposition. $(G_1 * G_2, \diamond)$ is a group.

2.1.16 Example. Here we give two examples of free product of groups.

- Let us consider two cyclic groups $G_1 = \langle a | a^n \rangle$ and $G_2 = \langle b | b^m \rangle$ respectively of order n and m. The free product $G_1 * G_2$ is given by the following:

$$G_1 * G_2 = \langle a, b | a^n, b^m \rangle.$$

- The free product of the free groups F_n and F_m respectively on n and m generators is the free group F_{n+m} on $n + m$ generators, that is,

$$F_n * F_m \cong F_{n+m}.$$

2.2 Fundamental Group

Here, we just try to give a brief overview on fundamental group. It is based of the notion of homotopy relation which involves some topological concepts such as continuous functions between two topological spaces. We are going to present the main definitions and results which lead to the notion of fundamental group. We conclude this section by stating the van Kampen theorem [4] which is one of the main tool needed for computing the fundamental group of several spaces.

In the sequel, X, Y and Z usually denote topological spaces and $I = [0, 1]$.

2.2.1 Definition (Homotopy Relation)**.** [12] Let $f, g : X \longrightarrow Y$ be continuous maps. the map f is said to be homotopic to g if there exists a continuous map $H : X \times I \longrightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, for all $x \in X$.

The map H is said to be a homotopy from f to g and we write $f \sim g$.

2.2.2 Proposition. [12] The homotopy relation "$\sim$" is an equivalence relation on the set of continuous maps from X to Y.

2.2.3 Proposition. [12] If we have $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$ continuous maps such that $f_1 \sim f_2$ and $g_1 \sim g_2$, then $g_1 f_1 \sim g_2 f_2$.

2.2.4 Definition (Homotopy Equivalence). [12] A continuous map $f : X \to Y$ is a homotopy equivalence if there exists a continuous map $g : X \to Y$ such that $gf \sim id_X$ and $fg \sim id_Y$.

2.2.5 Definition (Homotopy Equivalent Spaces). [12] If there exists a homotopy equivalence $f : X \to Y$, then X and Y are said to be homotopy equivalent or to have the same homotopy type. In this case we write $X \simeq Y$.

2.2.6 Definition (Path). [12] A path in X is just a continuous map $\alpha : I \longrightarrow X$.

2.2.7 Definition (Composition of Paths). [12] Let α and β be paths in X verifying $\alpha(1) = \beta(0)$. The composition of paths α and β, denoted $\alpha \cdot \beta$, is a path given by the following construction:

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s) & \text{if} \quad 0 \le s \le \dfrac{1}{2}; \\ \beta(2s-1) & \text{if} \quad \dfrac{1}{2} \le s \le 1. \end{cases} \tag{2.2.1}$$

2.2.8 Definition (Homotopy Relation Between Paths). [12] Let α and β be paths in X such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. The path α is said to be homotopic to β and we write $\alpha \sim \beta$, if there exists a continuous map $H : I \times I \to X$ such that:

(H1) $H(s,0) = \alpha(s)$ and $H(s,1) = \beta(s)$, for all $s \in I$;

(H2) $H(0,t) = \alpha(0)$ and $H(1,t) = \alpha(1)$, for all $t \in I$;

Here, H is called a homotopy from α to β.

2.2.9 Proposition. [12] Let $x, y \in X$. The homotopy relation "$\sim$" defined above is an equivalence relation on the set of paths α such that $\alpha(0) = x$ and $\alpha(1) = y$.

2.2.10 Notation. The equivalence class of a path α, with respect to the equivalence relation "$\sim$", is denoted $[\alpha]$ and we then have the following:

$$\alpha \sim \beta \text{ if and only if } [\alpha] = [\beta].$$

2.2.11 Definition (Constant Path). [12] Let $x \in X$. The constant path at x is the path denoted by c_x and defined as $c_x(s) = x$, $\forall s \in I$.

2.2.12 Definition (Inverse Path). [12] Let α a path. The inverse path of α is the path denoted by $\overline{\alpha}$ and defined as $\overline{\alpha}(s) = \alpha(1 - s)$, $\forall s \in I$.

2.2.13 Definition (Loop). [12] A loop in X is a path α such that $\alpha(0) = \alpha(1)$.

2.2.14 Notation. Let α be a loop and set $x = \alpha(0) = \alpha(1)$. We say that α is a loop at x. The set of equivalence classes $[\alpha]$, with respect to the equivalence relation "$\sim$", of loops at x is denoted by $\pi_1(X, x)$, that is;

$$\pi_1(X, x) = \{[\alpha] \mid x = \alpha(0) = \alpha(1)\}. \tag{2.2.2}$$

2.2.15 Proposition. [12] The set $\pi_1(X, x)$ has a group structure with respect to the internal operation given by:
$$[\alpha][\beta] = [\alpha \cdot \beta], \text{ for all } \alpha, \beta \in \pi_1(X, x). \tag{2.2.3}$$

The proof of this result is based on the four following lemmas.

2.2.16 Lemma. [12] Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ four paths such that $\alpha_1(0) = \alpha_2(0), \alpha_1(1) = \alpha_2(1) = \beta_1(0) = \beta_2(0)$ and $\beta_1(1) = \beta_2(1)$. If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$, then $\alpha_1 \cdot \beta_1 \sim \alpha_2 \cdot \beta_2$.

2.2.17 Lemma. [12] Let $x, y \in X$. If α is a path such that $\alpha(0) = x$ and $\alpha(1) = y$, then $\alpha \sim c_x \cdot \alpha$ and $\alpha \sim \alpha \cdot c_y$.

2.2.18 Lemma. [12] Let $x, y \in X$. If α is a path such that $\alpha(0) = x$ and $\alpha(1) = y$, then $\alpha \cdot \overline{\alpha} \sim c_x$ and $\overline{\alpha} \cdot \alpha \sim c_y$.

2.2.19 Lemma. [12] If α, β, γ are paths such that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then $\alpha \cdot (\beta \cdot \gamma) \sim (\alpha \cdot \beta) \cdot \gamma$, that is, the composition of paths is associative up to homotopy.

Proof. (Proposition 2.2.15). There are four things to check. Let $x \in X$.

- Well-defineness of the internal operation.

  Let $[\alpha_1], [\alpha_2], [\beta_1], [\beta_2] \in \pi_1(X, x)$ such that $[\alpha_1] = [\alpha_2]$ and $[\beta_1] = [\beta_2]$. We have that:

  $[\alpha_1] = [\alpha_2] \Rightarrow \alpha_1 \sim \alpha_2$ and $[\beta_1] = [\beta_2] \Rightarrow \beta_1 \sim \beta_2$, then from Lemma 2.2.16, $\alpha_1 \cdot \beta_1 \sim \alpha_2 \cdot \beta_2$. Therefore $[\alpha_1 \cdot \beta_1] = [\alpha_2 \cdot \beta_2]$.

- Existence of the identity element.

  The identity element in $\pi_1(X, x)$ for the considering internal operation is the constant path $[c_x]$. Indeed, let $\alpha \in \pi_1(X, x)$ we have $[\alpha][c_x] = [\alpha \cdot c_x]$ by definition of c_x. But from Lemma 2.2.17, we have $\alpha \cdot c_x \sim \alpha$. Therefore $[\alpha][c_x] = [\alpha]$.

  Analogously, we have $[c_x][\alpha] = [\alpha]$ by applying the same lemma as before.

- Inverse of an element.

  Let $[\alpha] \in \pi_1(X, x)$. The inverse element of $[\alpha]$ in $\pi_1(X, x)$ is the equivalent class $[\overline{\alpha}]$ of the inverse path $\overline{\alpha}$ (inverse loop in this case) of α. Indeed, from Lemma 2.2.18, we have $\alpha(0) = \alpha(1) = x \Rightarrow \alpha \cdot \overline{\alpha} \sim c_x$ and $\overline{\alpha} \cdot \alpha \sim c_x$. Therefore $[\alpha \cdot \overline{\alpha}] = [\overline{\alpha} \cdot \alpha] = [c_x]$.

- Associativity of the internal operation.

  Let $[\alpha], [\beta], [\gamma] \in \pi_1(X, x)$. We have the following:

$$\begin{aligned}
[\alpha]([\beta][\gamma]) &= [\alpha]([\beta \cdot \gamma]) \\
&= [\alpha \cdot (\beta \cdot \gamma)] \\
&= [(\alpha \cdot \beta) \cdot \gamma], \text{ by Lemma 2.2.19}; \\
&= [\alpha \cdot \beta][\gamma] \\
&= ([\alpha][\beta])[\gamma].
\end{aligned}$$

Therefore $[\alpha]([\beta][\gamma]) = ([\alpha][\beta])[\gamma]$.

Since the axioms of a group are verified, we have the desired result.   □

2.2.20 Definition (Fundamental Group). [12] Let $x \in X$. The fundamental group of X at x is the one from Proposition 2.2.15, that is, the group $\pi_1(X, x)$.

2.2.21 Definition (Path-connected Space). [12] Let X be a topology space. X is said to be path-connected if for every $x, y \in X$, there exists a path $\delta : I \to X$ verifying $\delta(0) = x$ and $\delta(1) = y$.

2.2.22 Theorem. *[2] If X is a path-connected space and $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. (Theorem 2.2.22). Let X be a path-connected space, $x_0, x_1 \in X$ and let δ be the path from x_0 to x_1 with inverse path $\overline{\delta}$ from x_1 back to x_0. Let α be a loop based at x_0, $\overline{\delta} \cdot \alpha \cdot \delta$ is then a loop based at x_1.

Consider the map $h_\delta : \pi_1(X, x_0) \to \pi_1(X, x_1)$ defined by $[\beta] \mapsto [\overline{\delta} \cdot \beta \cdot \delta]$. Let $H : I \times I \to X$ be a homotopy of loops α and β at x_0, for a fixed $t \in I$, H_t is also a loop at x_0 (where $H_t(s) = H(s, t)$). Therefore, $\overline{\delta} \cdot H_t \cdot \delta$ is well-defined and is a homotopy of loops $\overline{\delta} \cdot \alpha \cdot \delta$ and $\overline{\delta} \cdot \beta \cdot \delta$ at x_1. Thus h_δ is well-defined.

We have $h_\delta[\alpha \cdot \beta] = [\overline{\delta} \cdot \alpha \cdot \beta \cdot \delta] = h_\delta[\alpha] \cdot h_\delta[\beta]$, so h_δ is a homomorphism.

Consider the map $h_{\overline{\delta}}$, we have $h_{\overline{\delta}} h_\delta[\alpha] = h_\delta[\delta \cdot \alpha \cdot \overline{\delta}] = [\overline{\delta} \cdot \delta \cdot \alpha \cdot \delta \cdot \overline{\delta}] = [\alpha]$ and similarly, $h_\delta h_{\overline{\delta}}[\beta] = [\beta]$. So h_δ is a bijection.

<u>Conclusion:</u> We can conclude that $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.   □

2.2.23 Proposition. [3] The fundamental group of the circle S^1 is given by $\pi_1(S^1) \cong \mathbb{Z}$.

2.2.24 Proposition. [12] If $f : (X, x_0) \longrightarrow (Y, y_0)$ be a pointed map (that is, f is a continuous map such that $f(x_0) = y_0$), then the map

$$\pi_1(f) = f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$[\alpha] \longmapsto [f \circ \alpha] \tag{2.2.4}$$

is a homomorphism of groups called induced homomorphism.

2.2.25 Proposition. [12] If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$ is an isomorphism, for all $x \in X$.

The notion of deformation retract will be very useful in the following to compute fundamental group of some spaces. We will use the fact that two spaces have isomorphic fundamental groups if one deformation retracts onto the other (See Proposition 2.2.27)

2.2.26 Definition (Deformation Retract). [12] Let A be a subspace of a space X. A is said to be a retract of X (or X retracts onto A) if there exists a continuous map $r : X \to A$ such that $r \circ i = id_A$. If in addition $i \circ r \sim id_X$, then A is said to be a deformation retract of X (or X deformation retracts onto A). Here, $i : A \hookrightarrow X$ is the inclusion map.

2.2.27 Proposition. [12] Let us consider the inclusion map $i : A \hookrightarrow X$. If X deformation retracts onto A, then the induced homomorphism i_* is an isomorphism, that is, $\pi_1(A, x) \cong \pi_1(X, x)$, for all $x \in A$.

Van Kampen's Theorem

The following van Kampen theorem is used for computations of the fundamental group of spaces which are constructed out of simpler ones (cell complexes for example). It expresses the structure of the fundamental group of a topological space X in terms of the fundamental groups of opens, path-connected subspaces A_α that cover X.

2.2.28 Theorem (Van Kampen). *[4] Let X be a topological space and let $x_0 \in X$ be a basepoint.*

1. *Assume $X = \cup_\alpha A_\alpha$ and that, for all α:*

   (a) *A_α is open;*

   (b) *A_α is path-connected;*

   (c) *$x_0 \in A_\alpha$.*

   Then, the homomorphism $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ is surjective.
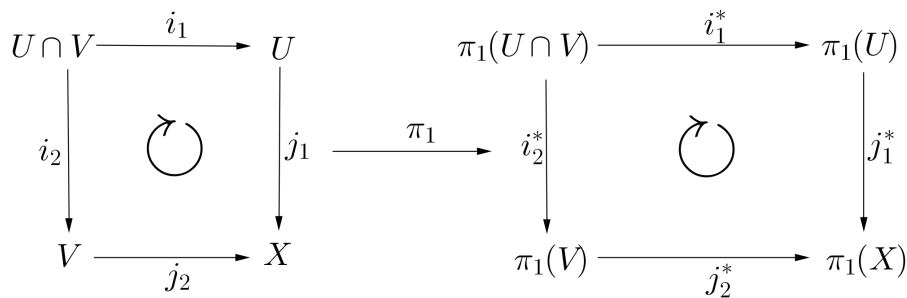
2. *Assume now we also have that each three-way intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected. Also, define the following homomorphisms:*

   (a) *$i_\alpha^* : \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusion map $i_\alpha : A_\alpha \hookrightarrow X$;*

   (b) *$j_{\alpha,\beta}^* : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ induced by the inclusion map $j_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$.*

Then the kernel of Φ is the normal subgroup N generated by all elements in $*_\alpha \pi_1(A_\alpha)$ of the form, $j_{\alpha\beta}(\omega)j_{\beta\alpha}(\omega)^{-1}$, where $\omega \in A_\alpha \cap A_\beta$. In this case, Φ induces an isomorphism $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$.

2.2.29 Remark. [12] If the cover $\{A_\alpha\}_\alpha$ is reduced to two subsets U, V of X, then van Kampen's theorem also reduces to the following:

If U, V and $U \cap V$ are path connected, $x_0 \in U \cap V$ and $X = U \cup V$, then the following commutative diagrams hold:

$$
\begin{array}{ccc}
U \cap V \xrightarrow{\ i_1\ } U & \quad & \pi_1(U \cap V) \xrightarrow{\ i_1^*\ } \pi_1(U) \\
\downarrow{\scriptstyle i_2} \quad \circlearrowleft \quad \downarrow{\scriptstyle j_1} \xrightarrow{\ \pi_1\ } & & \downarrow{\scriptstyle i_2^*} \quad \circlearrowleft \quad \downarrow{\scriptstyle j_1^*} \\
V \xrightarrow{\ j_2\ } X & & \pi_1(V) \xrightarrow{\ j_2^*\ } \pi_1(X)
\end{array}
$$

and then

(i) there exists a map $\varphi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$ which is surjective.

(ii) $ker(\varphi)$ is a normal group N generated by $i_1^*(g)i_2^*(g^{-1}) \in \pi_1(U) * \pi_1(V)$ with $g \in \pi_1(U \cap V)$.

The points (i) and (ii) imply that $\pi_1(X) \cong (\pi_1(U) * \pi_1(V))/N$.

Fundamental Groups of some Spaces

Here we apply van Kampen's theorem to compute the fundamental group of what is called wedge sum of circles. We begin by the a general definition of the notion of wedge sum of spaces.

2.2.30 Definition (Wedge Sum). [4]

1. Let X and Y be two topological spaces. Let $x \in X$ and $y \in Y$ be basepoints respectively of X and Y. The wedge sum of X and Y (with respect to x and y) is the quotient $X \vee Y = X \amalg Y / \sim$ of the disjoint union $X \amalg Y$ by the smallest relation $\sim$ identifying x and y to a single basepoint.

2. Let $\{X_\alpha\}$ be a collection of topological spaces and $\{x_\alpha\}$ a collection of basepoints. The wedge sum of $\{X_\alpha\}$ is the quotient $\vee_\alpha X_\alpha = \amalg_\alpha X_\alpha / \sim$ of the disjoint union $\amalg_\alpha X_\alpha$ by the smallest relation $\sim$ identifying the collection $\{x_\alpha\}$ to a single basepoint.

2.2.31 Example. Computation of $\pi_1(\vee_\alpha S_\alpha^1)$.

Let $a_\alpha \subset S_\alpha^1$ be an open arc containing the common basepoint x_0 for each α. Let us consider a neighborhood A_α of S_α^1 given by $A_\alpha = S_\alpha^1 \vee_{\beta \neq \alpha} a_\beta$. The intersection of two or more such neighborhoods is always simply $\vee_\alpha a_\alpha$, that is, $\cap_\alpha A_\alpha = \vee_\alpha a_\alpha$, which is of the same homotopy type as a point and therefore its fundamental group is trivial, that is, $\pi_1(\cap_\alpha A_\alpha) = \pi_1(\vee_\alpha a_\alpha) = 0$ (See Figure 2.2.31).
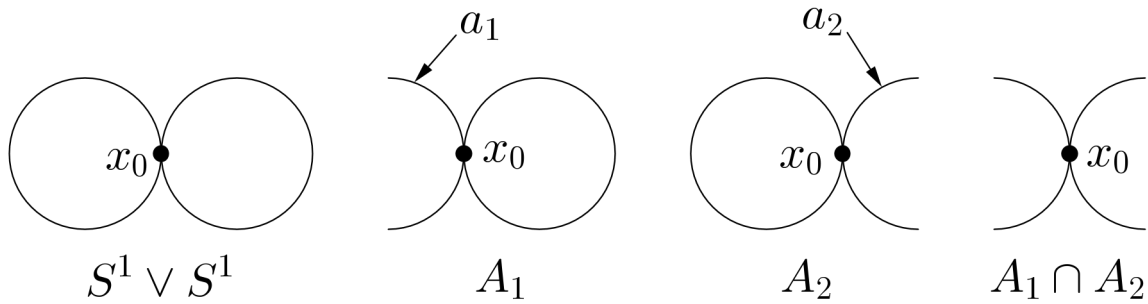

Figure 2.1: Wedge sum of two circles.

By van Kampen's theorem 2.2.28, it follows that we have the isomorphism:

$$\pi_1(\vee_\alpha S_\alpha^1) \cong *_\alpha \pi_1(S_\alpha^1) \cong *_\alpha \mathbb{Z}. \tag{2.2.5}$$

2.2.32 Example. Computation of $\pi_1(S^n)$ for all $n \geq 2$.

Let $U = S^n - \{(0, \cdots, 1)\}$ and $V = S^n - \{(0, \cdots, -1)\}$. $\{U, V\}$ is a cover of S^n and $U \cap V = S^n - \{(0, \cdots, 1), (0, \cdots, -1)\}$. U, V and $U \cap V$ are open subsets of S^n.

Considering the stereographic projection with respect to the north pole, we have that U is homeomorphic to $\mathbb{R}^n$ ($U \cong \mathbb{R}^n$), which implies that, $\pi_1(U) \cong \pi_1(\mathbb{R}^n)$ (See Figure 2.2). Similarly, we have $V \cong \mathbb{R}^n$ implies $\pi_1(V) \cong \pi_1(\mathbb{R}^n)$. But $\pi_1(\mathbb{R}^n) = 0$ so $\pi_1(U) \cong \pi_1(V) \cong \pi_1(\mathbb{R}^n) = 0$.

By van Kampen's theorem, $\pi_1(S^n) = \pi_1(U) * \pi_1(V) / N$, where N is the normal subgroup generated by $i_1^*(g) i_2^*(g^{-1}) \in \pi_1(U) * \pi_1(V)$ with $g \in \pi_1(U \cap V)$ (See Remark 2.2.29). Thus,
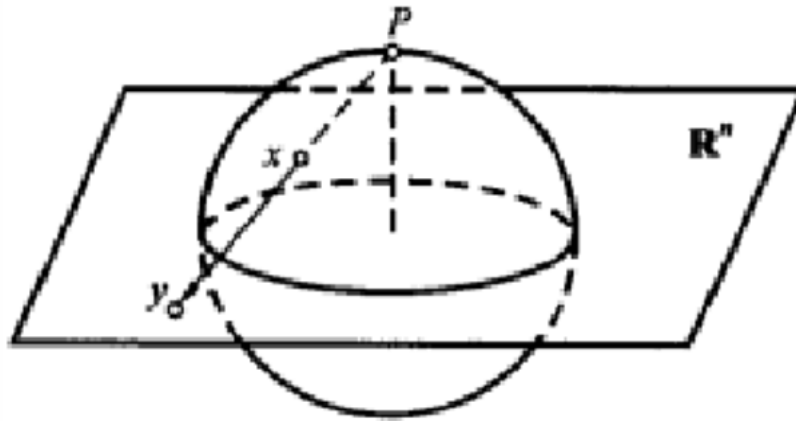
$$\pi_1(S^n) = 0. \tag{2.2.6}$$

Figure 2.2: Stereographic projection on U for $n = 2$ [11].

These results will be used in the second section of the next chapter to compute the fundamental group of closed and connected surfaces.

3. Fundamental Group of Cell Complexes

In this chapter, we first describe the topology of cell complexes. Then we state and prove a result which will allow us to compute easily the fundamental group of path-connected complexes. We end the chapter with the computation of fundamental group of both orientable and nonorientable closed and path-connected surfaces which are special cases of 2-dimensional cell complexes.

3.1 Topology of Cell Complexes

Many topological spaces of practical interest can be represented by a decomposition into subsets (cells), each with a simple topology, attached together along their boundaries. A decomposition of this form is commonly called a cell complex. We begin by the definition of a cell.

3.1.1 Definition (Cell). [4] A cell or an i-cell is a space of the following form:

$$e^i = \{x \in \mathbb{R}^i \mid \|x\| < 1\}, \ \forall i = 0, 1, 2, \cdots . \tag{3.1.1}$$

The integer i stands for the dimension of the cell.

3.1.2 Remark. By definition, we have $e^i = D^i \backslash \partial D^i$.

To build a cell complex, we start by what we can call ingredients, which are all cells needed for the construction. Notice that more than one cell, of a given dimension, can be required to construct a cell complex.

3.1.3 Definition (Cell Complex). [4] A cell complex is a space X constructed in the following way:

1. Consider the set of all 0-cells e^0 (points) among the ingredients and denote it X^0;

2. Build the 1-skeleton X^1 by attaching boundary of all 1-cells e^1 from ingredients to elements of X^0;

3. Build the n-skeleton X^n inductively by attaching boundary of all n-cells e^n from ingredients to X^{n-1}, that is;
$$X^n = \frac{X^{n-1} \amalg_\alpha D^n_\alpha}{x \sim \varphi_\alpha(x)}, \tag{3.1.2}$$
where $\varphi_\alpha : S^{n-1} \to X^{n-1}$ is a collection of maps.

4. Set $X = \cup_n X^n$ endowed with the weak topology, that is, a set A is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

3.1.4 Remark. If X is finite-dimensional, that is, $X = X^n$, then the topology is reduced to the usual one, that is the quotient topology.

3.1.5 Definition (Characteristic Map). [4] Each cell e^n_α is associated to a continuous map Φ_α called characteristic map and defined as the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \to X^n \hookrightarrow X, \tag{3.1.3}$$

where $\sigma : X^{n-1} \amalg_\alpha D_\alpha^n \to X^n$ is defined as follows:

$$\begin{cases} \sigma|_{X^{n-1}\amalg_\alpha(D_\alpha^n\backslash\partial D_\alpha^n)} = id \\ \sigma|_{\partial D_\alpha^n} = \varphi_\alpha. \end{cases}$$

3.1.6 Remark. [4] Another description of the weak topology mentioned in part 4 of Definition 3.1.3 is related to the notion of characteristic map. A subset $A \subset X$ of X is open (or closed) iff $\Phi_\alpha^{-1}(A)$ is open (or closed) in D_α^n for each characteristic map Φ_α.

3.1.7 Definition (Subcomplex). [4] Let X be a cell complex. $A \subset X$ is a subcomplex of X if A is a union of cells of X such that the closure of each cell in A is contained in A.

3.1.8 Proposition. [4] A finite cell complex, that is, one with only finitely many cells, is compact.

The result above is based on the fact that attaching a single cell preserves compactness. The following result is a sort of converse of the above Proposition 3.1.8 and will help us to prove Corollary 3.2.2.

3.1.9 Proposition. [4] A compact subspace of a cell complex is contained in a finite subcomplex.

Proof. (Proposition 3.1.9) Let $C \subset X$ be a compact subspace of X. The proof is divided into two parts. We first show that C can meet only finitely many cells in X and secondly, we show that the union of those cells is contained in a finite subcomplex of X.

1. Assume by contradiction that there is a infinite sequence of points $x_i \in C$ all lying in distinct cells. Let $S = \{x_1, x_2, \cdots\}$ to be the set of all those points. We claim that S is closed in X. To prove the claim, we will proceed by induction on n:

   - $S \cap X^0$ is closed in X^0 because of the discrete topology of X^0.
   - Assume that $S \cap X^{n-1}$ is closed in X^{n-1} and let's show that $S \cap X^n$ is closed X^n.
     For each n-cell e_α^n of X, $\varphi_\alpha^{-1}(S) = \varphi_\alpha^{-1}(S \cap X^{n-1})$ is closed in ∂D_α^n as the pre-image of a closed set. Furthermore, $\Phi_\alpha^{-1}(S)$ consists of at most one more point in D_α^n, so $\Phi_\alpha^{-1}(S)$ is closed in D_α^n.

   Then $S \cap X^n$ is closed in X^n for each n, which implies that S is closed in X.

   Using the same argument, we show that any subset of S is closed, that is, S has the discrete topology. But S is compact as a closed subset of a compact subspace C. Therefore S must be finite, which is a contradiction. Hence C is contained in a finite union of cells, that is, $C \subset \cup_{n\alpha} e_\alpha^n$ finite.

2. Now, it remains to show that a finite union of cells is contained in a finite subcomplex of X. Since a union of finite subcomplexes is again a finite subcomplex, this reduces to show that a single cell e_α^n is contained in a finite subcomplex. The image of the attaching map φ_α for e_α^n is a compact as the image of a compact by a continuous map. So by induction on dimension this image is contained in a finite subcomplex $A_\alpha^n \subset X^{n-1}$. So e_α^n is contained in the finite subcomplex $A \cup e_\alpha^n$. Thus $\cup_{n\alpha} e_\alpha^n \subset \cup_{n\alpha}(A_\alpha^n \cup e_\alpha^n)$ finite subcomplex.

Conclusion: From 1 and 2, we conclude that $C \subset \cup_{n\alpha}(A_\alpha^n \cup e_\alpha^n)$ finite subcomplex, that is, the compact subspace C of the cell complex X is contained in a finite subcomplex $\cup_{n\alpha}(A_\alpha^n \cup e_\alpha^n)$. $\square$

Surfaces are special cases of cell complexes. In the following, we give some definitions and results related to them.

3.1.10 Definition (Surface). [15] A surface can be defined as a geometrical shape that resembles a deformed plane.

3.1.11 Definition (Orientable and Nonorientable Surfaces). [10] Let M be a connected surface, and let $\alpha : S^1 \to M$ be an injective loop.

- α is orientable if the complement $M \backslash \alpha(S^1)$ is not connected, in which case it has 2 components.

- α is nonorientable if the complement $M \backslash \alpha(S^1)$ is connected.

A surface M is orientable if every $\alpha : S^1 \to M$ is orientable. Otherwise, M is nonorientable.

3.1.12 Definition (Closed Surface). [15] A closed surface is a surface which is compact and without boundary.

3.1.13 Theorem (Classification theorem of closed surfaces). *[7] Every closed surfaces is homeomorphic to one of the following:*

* *The sphere;*

* *The connected sum of g tori, for $g \geq 1$;*

* *The connected sum of k real projective planes, for $k \geq 1$.*

The first two surfaces belong to the family of orientable surfaces. The number g is called the genus of the surface. The third surface is a nonorientable surface.

3.1.14 Example.   * The sphere and the torus are closed and orientable surfaces;

* The projective plane and the Klein bottle are closed and nonorientable surfaces;

* An open-disk is a non-closed surface.

3.1.15 Definition (Connected Sum). [10] The connected sum of two surfaces M and N, denoted $M \# N$, is obtained by removing a disk from each of them and gluing them together along the boundary components which result.

3.1.16 Example. The following Figure 3.1 shows the connected sum process from two torus (orientable surface of genus 1) to an orientable surface of genus 2.

3.1.17 Theorem. *[7] Every compact surface admits a polygonal presentation.*

3.2   Fundamental Group of Cell Complexes

The goal here is to see how the fundamental group $\pi_1(Y, x_0)$, of a space Y obtained from a space X by attaching cells, and the fundamental group $\pi_1(X, x_0)$ of X are related. Let X be a path-connected space and consider the space Y obtained from X by following the next three steps.
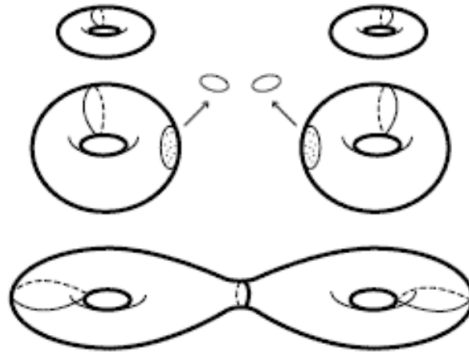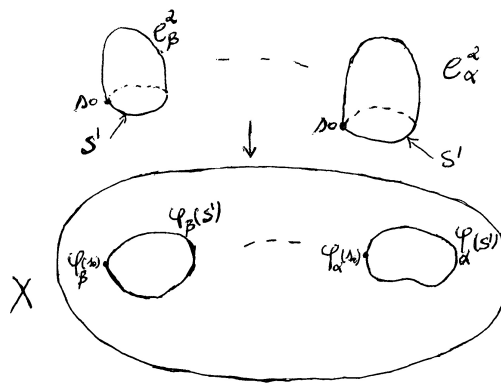
Figure 3.1: Connected sum of two torus [1].

Figure 3.2: Space X and collection of 2-cells.

Step 1. Let $\{e_\alpha^2\}_\alpha$ be a collection of 2-cells. Let $\varphi_\alpha : S^1 \to X$ be a continuous map, one for each α, and let $s_0 \in S^1$ be the basepoint (See Figure 3.2).

Step 2. After attaching the 2-cells e_α^2 to X via the maps $\varphi_\alpha : S^1 \to X$, we obtain the space Y (See Figure 3.3).

Step 3. Since s_0 is the basepoint of S^1, we have that φ_α determines a loop α at $\varphi_\alpha(s_0)$, that is, $\alpha(0) = \alpha(1) = \varphi_\alpha(s_0)$, with $\alpha = \varphi_\alpha \circ \beta$ where $\beta : I \to S^1$ is a loop at s_0. In the following we shall consider the map φ_α as the loop α. The basepoints $\varphi_\alpha(s_0)$ may not all coincide by varying α, but since we want to discuss about the fundamental groups of X and Y, we need to deal with one basepoint. This prompts us to choose a basepoint $x_0 \in X \subset Y$. Consider a path γ_α in X from x_0 to $\varphi_\alpha(s_0)$ for each α. Such a path exists since X is path-connected. Then, for each α, we have that $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is a loop at x_0 since X is a path-connect space (See Figure 3.3).

$$\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha(s) = \begin{cases} \gamma_\alpha(2s), & \text{if } 0 \leq s \leq \dfrac{1}{2} \\[2mm] \varphi_\alpha(4s-2), & \text{if } \dfrac{1}{2} \leq s \leq \dfrac{3}{4} \\[2mm] \gamma_\alpha(4-4s), & \text{if } \dfrac{3}{4} \leq s \leq 1. \end{cases}$$
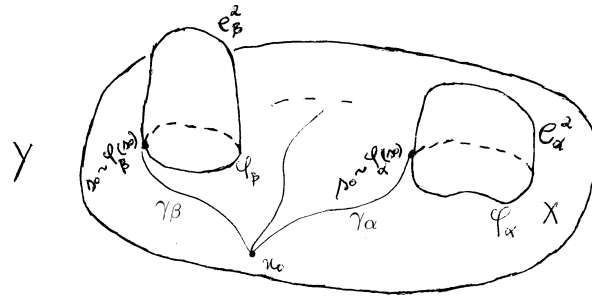
Figure 3.3: Space Y and loops $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ in Y at x_0.

Let N be the normal subgroup generated by all the loops $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ for varying α. The following theorem gives a relation between $\pi_1(Y, x_0)$ and $\pi_1(X, x_0)$.

3.2.1 Theorem. *[4] Let X be a path-connected space.*

1. *If Y is obtained from X by attaching 2-cells as described above, then the inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is N. Thus $\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N$.*

2. *If Y is obtained from X by attaching n-cells for a fixed $n > 2$, then the inclusion $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X, x_0) \cong \pi_1(Y, x_0)$.*

Proof. (Theorem 3.2.1)

Part 1. The proof is subdivided into three main steps. We begin with the construction of a space Z which deformation retracts onto Y and is more suitable for the application of van Kampen's Theorem. Then we apply van Kampen's theorem twice, firstly to a cover $\{A, B\}$ of Z and secondly to a cover $\{A_\alpha\}_\alpha$ of $A \cap B$.

- Construction of the space Z from Y.

  The space Z is obtained from Y by attaching rectangular bands $S_\alpha = I \times I \not\subseteq Y$ defined such that its bottom edge $I \times \{0\}$ is attached along the path γ_α, the right edge $1 \times I$ is attached along an arc $\beta_\alpha : I \to Y$ whose origin is at $\varphi_\alpha(s_0)$ and follows radially the shape of the 2-cell e_α^2, and all the left edges $\{0\} \times I$ of the different bands identified together. The top edges of the bands are not attached to anything. So the space Z is of the form

  $$Z = \frac{Y \amalg \left(\cup_\alpha(\gamma_\alpha(I) \times I)\right)}{(\gamma_\alpha(s), 0) \sim \gamma_\alpha(s), (\gamma_\alpha(1), t) \sim \beta_\alpha(t)}. \tag{3.2.1}$$

  We claim that Z deformation retracts onto Y. Indeed, consider the map $r : Z \to Y$ as follows:

  $$r(z) = \begin{cases} id_Y(z) & \text{if } z \in Y; \\ \gamma_\alpha(s_1) & \text{if } z = [(\gamma_\alpha(s_1), s_2)], \text{ with } s_1 \neq 1 \text{ and } s_2 \neq 0. \end{cases}$$

  * r is well-defined since $r[(\gamma_\alpha(s_1), 0)] = \gamma_\alpha(s_1) = z = id_Y(\gamma_\alpha(s_1))$, for $z = [(\gamma_\alpha(s_1), 0)]$ and also, $r[(\gamma_\alpha(0), s_2)] = \gamma_\alpha(0) = x_0 = \gamma_\beta(0) = r[(\gamma_\beta(0), s_2)]$, for $[(\gamma_\alpha(0), s_2)] = [(\gamma_\beta(0), s_2)]$. $r[(\gamma_\alpha(1), t)] = [(\gamma_\alpha(1), t)] = [\beta_\alpha(t)] = r[\beta_\alpha(t)]$ since $\beta_\alpha(t) \in Y$.
  * r is continuous since, assuming that W is an open subset of Y, then $r^{-1}(W) = W \cup_\alpha (\gamma_\alpha^{-1}(W) \times I)$ is open as union of open sets.
  * Consider the inclusion $i : Y \hookrightarrow Z$, we have $r \circ i(y) = r([y]) = [y] = y = id_Y(y)$ by definition of the map r. So $r \circ i = id_Y$.
  * We have $i \circ r(z) = \begin{cases} id_Y(z) & \text{if } z \in Y \\ (\gamma_\alpha(s_1), 0) & \text{otherwise} \end{cases}$.
  Consider the map $H : Z \times I \to Z$ defined by

$$H(z, t) = \begin{cases} [id_Y(z)] & \text{if } z \in Y \\ [(\gamma_\alpha(s_1), ts_2)] & \text{otherwise (where } z = (\gamma_\alpha(s_1), s_2)) \end{cases}.$$

  · H is well-defined since $H([(\gamma_\alpha(s_1), 0)], t) = [\gamma_\alpha(s_1)] = z = [id_Y(\gamma_\alpha(s_1))]$ by construction of Z, for $z = [(\gamma_\alpha(s_1), 0)]$ and also, $H([(\gamma_\alpha(0), s_2)], t) = [(\gamma_\alpha(0), t)] = [(x_0, t)] = [(\gamma_\beta(0), t)] = H([(\gamma_\beta(0), s_2)], t)$, for $[(\gamma_\alpha(0), s_2)] = [(\gamma_\beta(0), s_2)]$.
  · H is continuous since, assume O is an open subset of Z, then $H^{-1}(O) = (O \cup_\alpha (\gamma_\alpha^{-1}(O) \times I)) \times I$ is open as cartesian product of the segment I and union of open set.
  · $H(z, 0) = \begin{cases} id_Y(z) & \text{if } z \in Y \\ [(\gamma_\alpha(s_1), 0)] & \text{otherwise} \end{cases}$. So $H(z, 0) = i \circ r(z)$.
  $H(z, 1) = \begin{cases} id_Y(z) & \text{if } z \in Y \\ [(\gamma_\alpha(s_1), s_2)] & \text{otherwise} \end{cases}$. So $H(z, 1) = id_Z(z)$.

Therefore, we have $\pi_1(Y) \cong \pi_1(Z)$ by Proposition 2.2.27.

* Application of van Kampen's theorem to Z.

Let us now consider a point y_α on the 2-cell e_α^2 such that $y_\alpha \notin S_\alpha$. Set $A = Z - \cup_\alpha \{y_\alpha\}$ and $B = Z - X = \cup_\alpha \left[(e_\alpha^2 \cup S_\alpha) - \gamma_\alpha \cdot \varphi \cdot \overline{\gamma}_\alpha(I) \right]$. We have $A \cap B = (Z - X) - \cup_\alpha \{y_\alpha\} = \cup_\alpha \left[(e_\alpha^2 \cup S_\alpha) - (\{y_\alpha\} \cup \gamma_\alpha \cdot \varphi \cdot \overline{\gamma}_\alpha(I)) \right]$. Hence, A is path-connected (See Figure 3.4) and open since $A^C = \cup_\alpha \{y_\alpha\}$ is closed as a finite set, B is also path-connected (See Figure 3.4) and open and finally $A \cap B$ is also path-connected (See Figure 3.4) and open.

Since $x_0 \notin A \cap B$, to apply van Kampen's theorem to the cover $\{A, B\}$ of Z, we need to have a new basepoint, say $z_0 \in A \cap B$. That point z_0 is chosen close to x_0 on the segment (left edge on Figure 3.4) where all the bands S_α intersect.

Now, the hypothesis of van Kampen's theorem are verified. Consider the diagram as shown by Figure 3.5. Then

 (i) the map $\Phi : \pi_1(A, z_0) * \pi_1(B, z_0) \to \pi_1(Z, z_0)$ is surjective.
(ii) Its kernel $ker\Phi = N \triangleleft \pi_1(A, z_0) * \pi_1(B, z_0)$ generated by $i_1^*([\delta_\alpha])i_2^*([\delta_\alpha]^{-1}) \in \pi_1(A, z_0) * \pi_1(B, z_0)$, $\forall [\delta_\alpha] \in \pi_1(A \cap B)$ implies $N = \langle\langle \{[\delta_\alpha]\}_\alpha \rangle\rangle$.

(i) and (ii) imply that $\pi_1(Z, z_0) \cong (\pi_1(A, z_0) * \pi_1(B, z_0))/N$. But B is contractible (See Figure 3.4), that is, $\pi_1(B, z_0) = 0$, so we have

$$\pi_1(Z, z_0) \cong (\pi_1(A, z_0)/N. \tag{3.2.2}$$

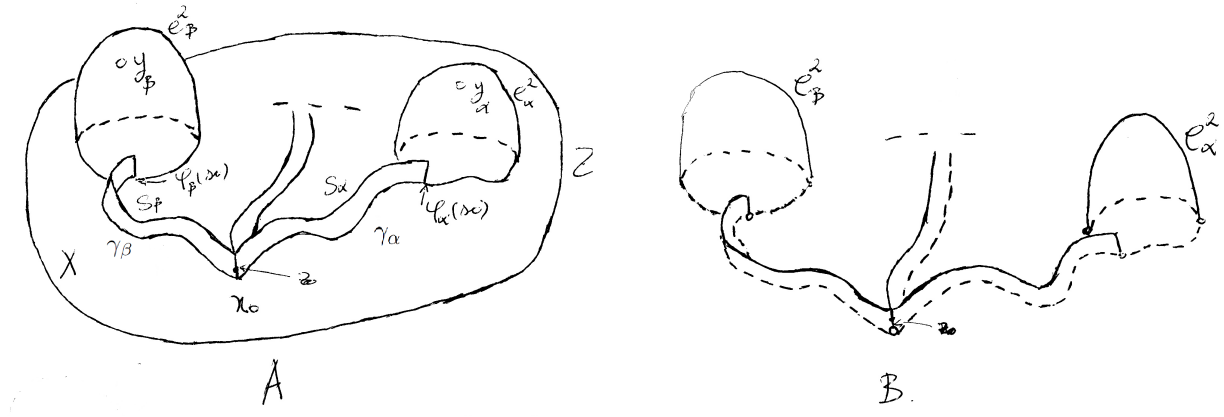To determine a more explicit expression of N, we need to compute $\pi_1(A \cap B)$.

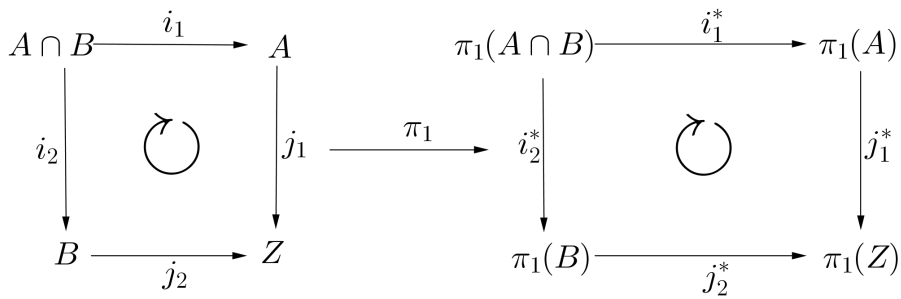Figure 3.4: Subspaces A and B of Z.

Figure 3.5: π_1 of a commutative square.

- Application of van Kampen's theorem to $A \cap B$.

  Here, for each α, we consider the open subsets $A_\alpha = A \cap B - \cup_{\beta \neq \alpha} e_\alpha^2$ and we apply van Kampen's theorem on the cover $\{A_\alpha | \alpha\}$ of $A \cap B$. We have that A_α's and $\cap_\alpha A_\alpha$ are open and path-connected, also $z_0 \in \cap_\alpha A_\alpha$ (See Figure 3.6). Applying van Kampen's theorem, we
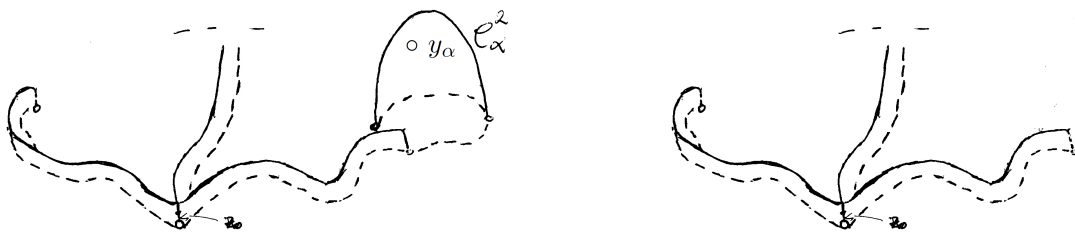

Figure 3.6: Spaces A_α (left) and $\cap_\alpha A_\alpha$ (Right).

have the following:

(i) the map $\Phi : *_\alpha \pi_1(A_\alpha, z_0) \to \pi_1(A \cap B, z_0)$ is surjective.

(ii) Its kernel $ker\Phi = N \triangleleft *_\alpha \pi_1(A_\alpha, z_0)$ generated by $i_1^*([\lambda_\alpha])i_2^*([\lambda_\alpha]^{-1}) \in *_\alpha \pi_1(A_\alpha, z_0)$, $\forall [\lambda_\alpha] \in \pi_1(\cap_\alpha A_\alpha)$ but $\cap_\alpha A_\alpha$ is contractible, that is, $\pi_1(\cap_\alpha A_\alpha, z_0) = 0$ (See Figure 3.6) which implies $N = 0$.

The points (i) and (ii) imply that $\pi_1(A \cap B, z_0) \cong *_\alpha \pi_1(A_\alpha, z_0)$. But each A_α deformation retracts onto the circle S^1 in $e_\alpha^2 - \{y_\alpha\}$ due to the holes created by the withdrawal of points y_α (See Figure 3.6), that is, $\pi_1(A_\alpha, z_0)) = \mathbb{Z}$ for each α, so we have $\pi_1(A \cap B, z_0) \cong *_\alpha \mathbb{Z}$. We have that $A = \dfrac{(Y - \cup_\alpha \{y_\alpha\}) \amalg (\cup_\alpha (\gamma_\alpha(I) \times I))}{(\gamma_\alpha(s), 0) \sim \gamma_\alpha(s)}$ deformation retracts onto X as in the case of Z and Y.

<u>Conclusion:</u> We have $\pi_1(X, x_0) \cong \pi_1(X, x_1) \cong \pi_1(A, x_1) \cong \pi_1(A, z_0)$ and $\pi_1(Y, z_0) \cong \pi_1(Y, x_0)$ since the fundamental group is independent of the choice of the basepoint for path-connected spaces (See Theorem 2.2.22). Therefore we have the following result from (3.2.2):

$$\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N, \text{ with } N = \langle\langle\{[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha]\}_\alpha\rangle\rangle, \tag{3.2.3}$$

Where $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha] = \Psi([\delta_\alpha])$, $\Psi : \pi_1(Z, z_0) \to \pi_1(Z, x_0)$ is the basepoint-change isomorphism.

Part 2. Now, assume that Y is obtained from X by attaching n-cells e_α^n for a fixed $n > 2$ and let us show that $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X, x_0) \cong \pi_1(Y, x_0)$.

Here we follow the same three steps as before:

- Construction of the space Z from Y.

  We make a similar construction of Z as in Part 1 and we also have that Z deformation retracts onto Y.

- Application of van Kampen's theorem to Z.

  Let us now apply the van Kampen's theorem to the cover $\{A, B\}$ of Z, where $A = Z - \cup_\alpha \{y_\alpha\}$ and $B = Z - X = \cup_\alpha [(e_\alpha^n \cup S_\alpha) - \gamma_\alpha \cdot \varphi \cdot \overline{\gamma}_\alpha(I)]$. We have $A \cap B = (Z - X) - \cup_\alpha \{y_\alpha\} = \cup_\alpha [(e_\alpha^n \cup S_\alpha) - (\{y_\alpha\} \cup \gamma_\alpha \cdot \varphi \cdot \overline{\gamma}_\alpha(I))]$. Hence, A is path-connected and open since $A^C = \cup_\alpha \{y_\alpha\}$ is closed as a finite set, B is also path-connected and open and finally $A \cap B$ is also path-connected and open.

  We have $x_0 \notin A \cap B$ so we choose another basepoint $z_0 \in A \cap B$ as before.

  Now the hypothesis of van Kampen's theorem are verified and we have the diagram from Figure 3.5. Then

  (i) the map $\Phi : \pi_1(A, z_0) * \pi_1(B, z_0) \to \pi_1(Z, z_0)$ is surjective.

  (ii) Its kernel $ker\Phi = N \triangleleft \pi_1(A, z_0) * \pi_1(B, z_0)$ generated by $i_1^*([\delta_\alpha])i_2^*([\delta_\alpha]^{-1}) \in \pi_1(A, z_0) * \pi_1(B, z_0)$, $\forall [\delta_\alpha] \in \pi_1(A \cap B)$ implies $N = \langle\langle\{[\delta_\alpha] \mid \alpha\}\rangle\rangle$.

  (i) and (ii) imply that $\pi_1(Z, z_0) \cong (\pi_1(A, z_0) * \pi_1(B, z_0))/N$. But B is contractible (See Figure 3.4), that is, $\pi_1(B, z_0)) = 0$, so we have

$$\pi_1(Z, z_0) \cong (\pi_1(A, z_0)/N. \tag{3.2.4}$$

  To determine a more explicit expression of N, we need to compute $\pi_1(A \cap B)$.

- Application of van Kampen's theorem to $A \cap B$.

  Here, for each α, we consider the open subsets $A_\alpha = A \cap B - \cup_{\beta \neq \alpha} e_\alpha^n$ and we apply van Kampen's theorem to the cover $\{A_\alpha \mid \alpha\}$ of $A \cap B$. We have that A_α's and $\cap_\alpha A_\alpha$ are open and path-connected, also $z_0 \in \cap_\alpha A_\alpha$.

  Now the hypothesis of van Kampen's theorem are verified, we have the following:

(i) the map $\Phi : *_\alpha \pi_1(A_\alpha, z_0) \to \pi_1(A \cap B, z_0)$ is surjective.

(ii) Its kernel $ker\Phi = N \vartriangleleft *_\alpha \pi_1(A_\alpha, z_0)$ generated by $i_1^*([\lambda_\alpha])i_2^*([\lambda_\alpha]^{-1}) \in *_\alpha \pi_1(A_\alpha, z_0)$, $\forall [\lambda_\alpha] \in \pi_1(\cap_\alpha A_\alpha)$ but $\cap_\alpha A_\alpha$ is contractible, that is, $\pi_1(\cap_\alpha A_\alpha, z_0) = 0$ which implies $N = 0$.

(i) and (ii) imply that $\pi_1(A \cap B, z_0) \cong *_\alpha \pi_1(A_\alpha, z_0)$. But A_α's deformation retract onto the sphere S^{n-1} in $e_\alpha^n - \{y_\alpha\}$ due to the withdrawal of points y_α, that is, $\pi_1(A_\alpha, z_0)) = 0$ since $n > 2$ (From the point 2 of Proposition 2.2.23) for some α, so we have $\pi_1(A \cap B, z_0) \cong 0$ so $N = 0$.

<u>Conclusion:</u> As in Part 1, we have the following result from (3.2.4):

$$\pi_1(Y, x_0) \cong \pi_1(X, x_0). \tag{3.2.5}$$

This completes the proof of the theorem. $\square$

The following result, based on the above Theorem 3.2.1, states that the fundamental group of an n-dimensional $(n \geq 2)$ path-connected cell complex X is reduced to the fundamental group of its 2-skeleton X^2.

3.2.2 Corollary. If X is a path-connected cell complex, then the inclusion of the 2-skeleton $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.

Proof. (Corollary 3.2.2). Let X be a path-connected cell complex. Let us prove that the inclusion of the 2-skeleton $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.

Here, we distinguish two cases: the finite and non finite-dimensional cases. In the sequel, the basepoint x_0 is taken in X^2 $(x_0 \in X^2)$.

- Assume that X is finite-dimensional, that is, $X = X^n$.

  Let us do this by induction on the dimension $n > 2$ of X.

  - Since the case $n = 2$ is trivial, let us consider the case $n = 3$.
    We have $X = X^3$ and the inclusion $X^2 \hookrightarrow X$. From the construction [4] of the cell complex $X = X^3$, it is obtained from X^2 by attaching 3-cells e_α^3 to X^2 via the maps $\varphi_\alpha : S^2 \to X^3$. Therefore from Part 2, we know that the inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$ since $n = 3 > 2$.

  - Let us suppose that inclusion $X^2 \hookrightarrow X^{n-1}$ induces the isomorphism $\pi_1(X^2, x_0) \cong \pi_1(X^{n-1}, x_0)$ for $n > 2$ and let us prove that we have the same for $X = X^n$.
    From the construction of the cell complex $X = X^n$, it is obtained from X^{n-1} by attaching n-cells e_α^n to X^{n-1} via the maps $\varphi_\alpha : S^{n-1} \to X^n$. Therefore from Part 2, we know that the inclusion $X^{n-1} \hookrightarrow X^n$ induces an isomorphism $\pi_1(X^{n-1}, x_0) \cong \pi_1(X^n, x_0)$ since $n > 2$. But from induction hypothesis, we obtain $\pi_1(X^2, x_0) \cong \pi_1(X^n, x_0)$.

  Therefore, $\pi_1(X^2, x_0) \cong \pi_1(X^n, x_0)$, $\forall n \geq 2$.

- Now, assume that X is non finite-dimensional.

  Here, the idea is to show that $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$ is surjective and injective.

- *Surjection.*

  So let $\delta \in \pi_1(X, x_0)$, then its image $\delta(I)$ is compact as an image of a compact by a continuous map. Proposition 3.1.9 implies that there exists $n < \infty$ such that δ is a loop in X^n at x_0. But, as in the finite-dimensional case, from Part 2 we have that $\pi_1(X^2, x_0) \cong \pi_1(X^n, x_0)$, which implies that δ is homotopic to a loop in X^2 at x_0. Thus $\pi_1(X, x_0) \subseteq \pi_1(X^2, x_0)$, that is, $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$ is surjective.

- *Injection.*

  Let δ a loop in X^2 at x_0 which is nullhomotopic in X, that is, δ is an element of the kernel of the map $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$. Suppose that δ is nullhomotopic in X via a Homotopy $H : I \times I \to X$. Its image $H(I \times I)$ is compact as the image of a compact by a continuous map and as previously, by applying Proposition 3.1.9, we have that $\exists n < \infty$ such that δ is a loop in X^n at x_0. But from Part 2 we have that $\pi_1(X^2, x_0) \to \pi_1(X^n, x_0)$ is injective what implies δ is nullhomotopic in X^2. Thus $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$ is injective.

Thus $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$, for all path-connected cell complex X of infinite-dimension.

<u>Conclusion:</u> Therefore, $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$, for all path-connected cell complex X. $\qquad\square$

Fundamental Group of Closed and Connected Surfaces

The purpose is to compute the fundamental groups of orientable and nonorientable surfaces of genus g. These are supposed to be closed and path-connected surfaces.

The following theorem gives the expression of the fundamental group of an orientable surface of genus g.

3.2.3 Theorem. *If M_g is an orientable surface of genus g, then $\pi_1(M_g)$ has the presentation of the form:*

$$\pi_1(M_g) \cong \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \rangle. \tag{3.2.6}$$

Proof. (Theorem 3.2.3). Let M_g be an orientable surface of genus g. It is a 2-dimensional cell complex and for its construction, we need one 0-cell, $2g$ 1-cells and one 2-cell. Its 1-skeleton is constituted by the one 0-cell and the $2g$ 1-cells is a wedge sum of $2g$ circles S^1 (that is $\vee_{i=1}^{2g} S_i^1$) labeled $a_1, a_2, \cdots, a_g, b_1, b_2, \cdots, b_g$. An orientable surface M_g can be represented by a $4g$-gon [5] (Polygon of $4g$ edges, each edge representing a generator or its inverse) with boundary $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = [a_1, b_1][a_2, b_2] \cdots [a_g, b_g]$ (Product of commutators). The construction of the polygonal representation of M_g ended by attaching a 2-cell along the loop given by the former product of the commutators, resulting in an orientable surface of genus g (See Figure 3.7).

We know that π_1 of the wedge sum of $2g$ circles is the free group on $2g$ generators (See relation (2.2.5)), so $\pi_1(\vee_{i=1}^{2g} S_i^1) = \langle a_1, b_1, \cdots, a_g, b_g \mid \rangle$, the free group generated by $a_1, b_1, \cdots, a_g, b_g$. But according to the construction of M_g we have that $\pi_1(M_g) = \pi_1(\vee_{i=1}^{2g} S_i^1)/N$ from Part 1 of Theorem 3.2.1, then $\pi_1(M_g) = \langle a_1, b_1, \cdots, a_g, b_g \rangle/N$ where N is the normal subgroup of $\langle a_1, b_1, \cdots, a_g, b_g \mid \rangle$ generated by the word given by the product of commutators $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g]$. Therefore

$$\pi_1(M_g) \cong \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \rangle.$$
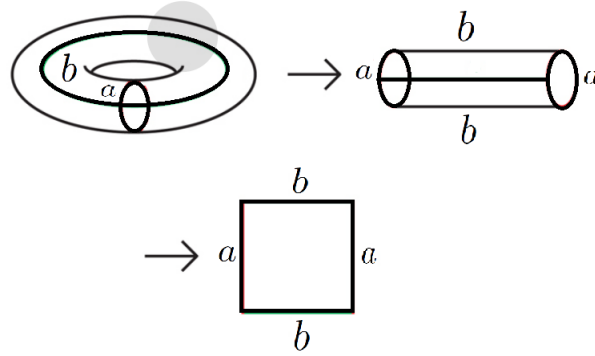
$\qquad\square$

Figure 3.7: Polygonal construction of the torus $T = M_1$.

Consider now the following theorem which gives the expression of the fundamental group of a nonorientable surface of genus g.

3.2.4 Theorem. *If N_g is an nonorientable surface of genus g, then $\pi_1(M_g)$ has the presentation of the form:*

$$\pi_1(N_g) \cong \langle a_1, \cdots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 \rangle. \tag{3.2.7}$$

Proof. (Theorem 3.2.4). Let N_g be a nonorientable surface of genus g. Like the g-genus orientable surface, N_g is a 2-dimensional cell complex and for its construction, it is needed one 0-cell, g 1-cells and one 2-cell. Its 1-skeleton, constituted by the one 0-cell and the g 1-cells, is a wedge sum of g circles S^1 (that is $\vee_{i=1}^g S_i^1$) labeled $a_1, a_2, \cdots, a_g$. A nonorientable surface N_g can be represented by a $2g$-gon [5] (Polygon of $2g$ edges, two edges representing a generator twice or a generator and its inverse) with boundary $a_1^2 a_2^2 \cdots a_g^2$ (Product of squares of generators). The construction of the polygonal representation of N_g end by attaching a 2-cell along the loop given by the former product of squares, resulting in a nonorientable surface of genus g.

Polygonal representations of projective plane and the Klein bottle are respectively given by Figures 3.8 and 3.9.
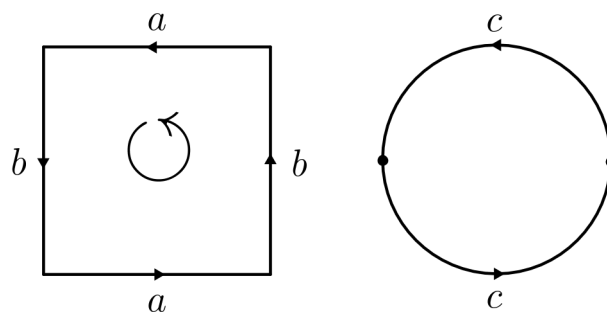

Figure 3.8: Polygonal representations of the projective plane $\mathbb{R}\mathbf{P}^2 = N_1$.

Case of N_2, the Klein bottle K.

We know that π_1 of the wedge sum of g circles is the free group on g generators (See relation (2.2.5)), so $\pi_1(\vee_{i=1}^g S_i^1) = \langle a_1, \cdots, a_g \mid \ \rangle$, the free group generated by $a_1, \cdots, a_g$. But according
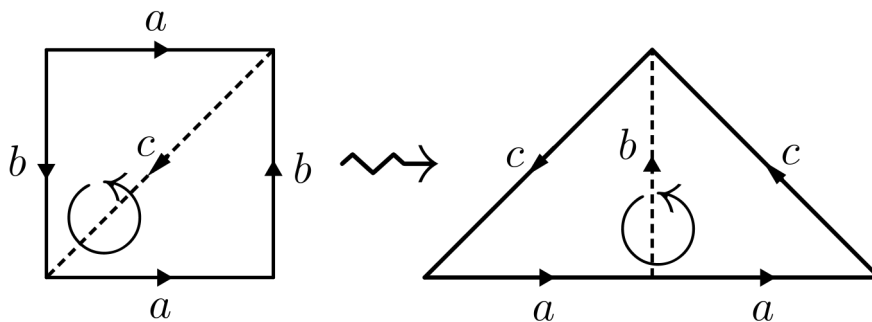
Figure 3.9: Polygonal representations of the Klein bottle $K = N_2$.

to the construction of N_g we have that $\pi_1(N_g) = \pi_1(\vee_{i=1}^g S_i^1)/N$ from Part 1 of Theorem 3.2.1, then $\pi_1(N_g) = \langle a_1, \cdots, a_g \rangle/N$ where $N = \langle\langle a_1^2 a_2^2 \cdots a_g^2 \rangle\rangle$ is the normal subgroup of $\langle a_1, a_2, \cdots, a_g \mid \rangle$ generated by the word given by the product of squares $a_1^2 a_2^2 \cdots a_g^2$. Therefore

$$\pi_1(N_g) \cong \langle a_1, \cdots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 \rangle.$$

$\square$

4. Applications of Fundamental Group of Cell Complexes

In this chapter, we investigate two main applications of the fundamental group to cell complexes. The first one says that if two surfaces are homotopy equivalent, then they have the same genus (See Theorem 4.1.1 and Theorem 4.1.2). The second one says that for a group G, there exists a 2-dimensional cell complex X_G such that $\pi_1(X_G) \cong G$ (See Theorem 4.2.1).

4.1 Homeomorphism of Closed and Path-connected Surfaces

We distinguish two cases, respectively the case of orientable and nonorientable surfaces.

- Case of orientable surfaces.

  4.1.1 Theorem. *[4] Let M_g be an orientable surface of genus g. Then M_g is not homeomorphic (or even homotopy equivalent) to M_h if $g \neq h$.*

  Proof. (Theorem 4.1.1)

  Let M_g and M_h be two orientable surfaces. Assume by contradiction that $M_g \simeq M_h$, that is, M_g and M_h are homotopy equivalent, and let us prove that $g = h$. Let us consider the group $Ab(\pi_1(M_g))$ the abelianization of the fundamental group $\pi_1(M_g)$ of the orientable surface M_g. From the expression of $\pi_1(M_g)$ computed in Theorem 4.1.1 and Definition 2.1.10 of abelianization, we have:

  $$\begin{aligned}
  Ab(\pi_1(M_g)) &= \langle a_1, b_1 \cdots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \rangle / \langle\langle \{[a, b] \mid a, b \in \pi_1(N_g)\} \rangle\rangle \\
  &= \langle a_1, b_1 \cdots, a_g, b_g \mid 1 \cdot 1 \cdots 1 \rangle \\
  &= \langle a_1, b_1 \cdots, a_g, b_g \mid \ \rangle, \text{ for } a = a_1 a_2 \cdots a_g \\
  &= *_{i=1}^g [\langle a_i \rangle * \langle b_i \rangle] \\
  &= *_{2g} \mathbb{Z}.
  \end{aligned}$$

  From hypothesis we have that $M_g \simeq M_h$, that is, $\pi_1(M_g) \cong \pi_1(M_h)$, hence the abelianization of these groups are isomorphic (that is, $*_{2g}\mathbb{Z} \cong *_{2h}\mathbb{Z} \Leftrightarrow 2g = 2h$), which implies $g = h$.

  <u>Conclusion:</u> Therefore given M_g and M_h orientable surfaces, we have $g \neq h \Rightarrow M_g$ is not homotopy equivalent to M_h and therefore $g \neq h \Rightarrow M_g$ is not homeomorphic to M_h. $\square$

- Case of orientable surfaces.

  4.1.2 Theorem. *[4] Let N_g be an nonorientable surface of genus g. Then N_g is not homeomorphic (or even homotopy equivalent) to N_h if $g \neq h$.*

  Proof. (Theorem 4.1.2).

  Let N_g and N_h be two nonorientable surfaces. Assume by contradiction that $N_g \simeq N_h$, that is, M_g and M_h are homotopy equivalent, and let us prove that $g = h$. Let us consider the group

$Ab(\pi_1(N_g))$ the abelianization of the fundamental group $\pi_1(N_g)$ of the nonorientable surface N_g. From the expression of $\pi_1(N_g)$ computed in Theorem 4.1.2 and Definition 2.1.10 of abelianization, we have:

$$
\begin{aligned}
Ab(\pi_1(N_g)) &= \langle a_1, a_2, \cdots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 \rangle / \langle\langle \{[a,b] \mid a,b \in \pi_1(N_g)\} \rangle\rangle \\
&= \langle a_1, a_2, \cdots, a_g \mid (a_1 a_2 \cdots a_g)^2 \rangle \\
&= \langle a_1, a_2, \cdots, a_{g-1}, a_g, a \mid a^2 \rangle, \text{ for } a = a_1 a_2 \cdots a_g \\
&= \langle a_1, a_2, \cdots, a_{g-1}, a \mid a^2 \rangle \\
&= \langle a \mid a^2 \rangle *_{i=1}^{g-1} \langle a_i \rangle \\
&= \mathbb{Z}_2 *_{g-1} \mathbb{Z}.
\end{aligned}
$$

From hypothesis we have that $N_g \simeq N_h$, that is, $\pi_1(N_g) \cong \pi_1(N_h)$, hence the abelinizations of these groups are isomorphic (that is, $\mathbb{Z}_2 *_{g-1} \mathbb{Z} \cong \mathbb{Z}_2 *_{h-1} \mathbb{Z} \Leftrightarrow g-1 = h-1$), which implies $g = h$.

Conclusion: Therefore given N_g and N_h orientable surfaces, we have $g \neq h \Rightarrow N_g$ is not homotopy equivalent to N_h and then $g \neq h \Rightarrow N_g$ is not homeomorphic to N_h. $\qquad\square$

4.2   From a Group to a Topological Space

Given a group G, can we construct a topological space X_G such that its fundamental group is isomorphic to G? The purpose of this part is to apply computation of fundamental group of cell complexes to answer this natural question. The answer is given by the following corollary of Theorem 3.2.1.

4.2.1 Theorem. _[4] For every group G, there is a 2-dimensional cell complex X_G with $\pi_1(X_G) \cong G$._

Proof. (Theorem 4.2.1).

This proof uses two tools, Theorem 3.2.1 and the construction of cell complexes.

Let G be a group and let us find a 2-dimensional cell complex X_G such that $\pi_1(X_G) \cong G$.

From Proposition 2.1.7, there exists elements g_α's and r_β's in G such that we have the following presentation:

$$G \cong \langle g_\alpha \mid r_\beta \rangle. \tag{4.2.1}$$

Let us consider the free group $\langle g_\alpha \mid \ \rangle$ generated by the g_α's and the normal subgroup $\langle\langle r_\beta \rangle\rangle$ of G generated by the r_α's. Then, relation (4.2.1) becomes:

$$G \cong \langle g_\alpha \mid \ \rangle / \langle\langle r_\beta \rangle\rangle. \tag{4.2.2}$$

Here $\langle\langle r_\beta \rangle\rangle = ker\theta'$, where the surjection $\theta' : \langle g_\alpha \mid \ \rangle \to G$ comes from Definition 2.1.3 of free group.

Let us consider a one skeleton $X^1 = \vee_\alpha S_\alpha^1$, which is the wedge sum of circles S_α^1 (obtained by gluing the boundaries of 1-cells e_α^1 to a 0-cell e^0), one for each g_α's. We know that:

$$\pi_1(X^1) \cong \langle g_\alpha \mid \ \rangle. \tag{4.2.3}$$

Now, let us consider the space X obtained from X^1 by attaching 2-cells e_β^2 along the loops associated to the words r_β. For instance if $G = \langle x, y | xy = yx \rangle$, then there are two generators $g_1 = x$ and $g_2 = y$ and one relator $r = xyx^{-1}y^{-1}$ for the relation $xyx^{-1}y^{-1} = 1$. So the associated loop wraps x then y then x^{-1} and finally y^{-1} from the basepoint x_0 (See Figure 4.1).
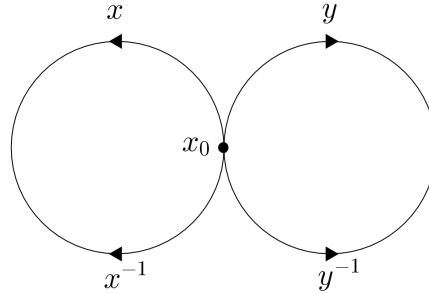

Figure 4.1: The 1-skeleton X^1 of X.

The space X is a 2-dimensional cell complex and from Part 1 of Theorem 3.2.1, the inclusion $X^1 \to X$ induces a surjection $\pi_1(X^1) \to \pi_1(X)$ whose kernel is N and we thus have $\pi_1(X) \cong \pi_1(X^1)/N$. But by construction of X, we have

$$\pi_1(X) \cong \langle g_\alpha \mid \rangle / \langle\langle r_\beta \rangle\rangle. \tag{4.2.4}$$

So by identification, we have $N = \langle\langle r_\beta \rangle\rangle$. Thus relations (4.2.2) and (4.2.4) give us

$$\pi_1(X) \cong G. \tag{4.2.5}$$

Conclusion: Therefore, it suffices to take $X_G = X$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Description of X_G.

The goal here is to describe the space X_G for $G = \langle a | a^n \rangle$ $(n \neq 0)$ considering the notations of Theorem 4.2.1. It is obtained by attaching a 2-cell on the circle S^1 via the map $\varphi : \mathbb{C} \supset S^1 \to S^1 \subset \mathbb{C}$ such that $\varphi(z) = z^n$. For the description, we need the following proposition which will give us a best idea of the space $X_G = \dfrac{D^2 \amalg S^1}{z \sim \varphi(z)}$.

4.2.2 Proposition. If we consider the following maps defined on the circle S^1 for $n \geq 1$,

$$\varphi : \partial D^2 \;\to\; S^1 \qquad\qquad \text{and} \qquad\qquad \psi : \partial D^2 \;\to\; \partial D^2$$
$$ z \;\mapsto\; z^n \qquad\qquad\qquad\qquad\qquad\qquad z \;\mapsto\; ze^{\frac{2\pi i}{n}} \;,$$

then we have the following homeomorphism:

$$\frac{D^2 \amalg S^1}{z \sim \varphi(z)} \;\cong\; \frac{D^2}{z \sim \psi(z)}. \tag{4.2.6}$$

Proof. (Proposition 4.2.2).

Let $X = \dfrac{D^2 \amalg S^1}{z \sim \varphi(z)}$ and $Y = \dfrac{D^2}{z \sim \psi(z)}$, where $z \in \partial D^2$.

We will define two maps $\Phi : X \to Y$ and $\Psi : Y \to X$ such that $\Phi\Psi = id_Y$ and $\Psi\Phi = id_X$. First, consider the map $p : S^1 \to S^1$, defined by $p(z) = z^n$. The map p is a non constant analytic function defined on S^1, then according to the Open Mapping Theorem for Analytic Functions [13], p is an open map. Also the map p is continuous as a polynomial function.

- Defining Φ. First define $f : D^2 \amalg S^1 \to Y$ as:

$$f(z) = \begin{cases} [z], & \text{if } z \in D^2 \backslash \partial D^2 \\ [e^{\frac{i\theta}{n}}], & \text{if } z = e^{i\theta} \in \partial D^2 \amalg S^1. \end{cases}$$

  Assume that $z \sim z'$. Thus without lost of generality, either $z = z' \in D^2 \backslash \partial D^2$ or $z \in \partial D^2$, $z' \in S^1$ and $z' = z^n = e^{n\theta i}$. f is well defined on $D^2 \backslash \partial D^2$. We have $f(z') = [e^{\frac{n\theta i}{n}}] = [e^{i\theta}] = [z] = f(z)$, therefore f is well-defined in the whole $D^2 \amalg S^1$. We claim that the map f is continuous. Indeed, let $U \subset Y$ open. We have:

$$f^{-1}(U) = (U \cap (D^2 \backslash \partial D^2)) \cup (p(U \cap S^1))$$

  is open as a union of open set since p is an open map.

  So the map f passes to the quotient and this gives rise to:

$$\Phi : X \to Y$$
$$[x] \mapsto f(x)$$

  By definition, Φ is continuous.

- Defining Ψ. First define $g : D^2 \to X$ as:

$$g(z) = \begin{cases} [z], & \text{if } z \in D^2 \backslash \partial D^2 \\ [z^n], & \text{if } z \in \partial D^2. \end{cases}$$

  The map g is well-defined D^2 by definition. We claim that the map g is continuous. Indeed, let $V \subset X$ open. We have:

$$g^{-1}(V) = (V \cap (D^2 \backslash \partial D^2)) \cup (p^{-1}(V \cap \partial D^2))$$

  is open as a union of open set since p is a continuous map.

  So the map g passes to the quotient and this gives rise to:

$$\Psi : Y \to X$$
$$[y] \mapsto g(y)$$

  By definition, Φ is continuous.

If $z \in D^2 \backslash \partial D^2$, then $\Phi\Psi([z]) = \Phi(g(z)) = \Phi([z]) = f(z) = [z] = id_Y([z])$ and if $z = e^{i\theta} \in \partial D^2$, then $\Phi\Psi([z]) = \Phi(g(z)) = \Phi([e^{in\theta}]) = f(e^{in\theta}) = [e^{\frac{in\theta}{n}}] = [e^{i\theta}] = [z] = id_Y([z])$.

If $z \in D^2 \backslash \partial D^2$, then $\Phi\Psi([z]) = \Phi(g(z)) = \Phi([z]) = [f(z)] = [z] = id_X([z])$ and if $z = e^{i\theta} \in \partial D^2 \amalg S^1$, then $\Phi\Psi([z]) = \Phi(g(z)) = \Phi([z^n]) = f(z^n) = f(e^{\frac{i\theta}{n}}) = [e^{\frac{in\theta}{n}}] = [e^{i\theta}] = [z] = id_X([z])$.

Therefore $\Phi : X \to Y$ is a homeomorphism between X and Y with $\Psi : Y \to X$ as its inverse homeomorphism.

Conclusion: $\dfrac{D^2 \amalg S^1}{z \sim \varphi(z)} \cong \dfrac{D^2}{z \sim \psi(z)}.$ $\square$

The description of X_G is done with respect to the value of n. We describe particular cases where $n = 1, 2, 3$ and we end with a generalization for an arbitrary value of n.

- Case $n = 1$.

  In this case we have $\varphi = id_{S^1}$, so from Proposition 4.2.2 the space $X_G = \dfrac{D^2 \amalg S^1}{z \sim z}$ is exactly the well known unite disk D^2.

- Case $n = 2$.

  As said above, the 1-skeleton of X_G is the circle S^1. The 2-cell $e^2 = D^2 \backslash \partial D^2$ is considered as divided into two equal parts, regions respectively determined by the oriented angles $(\widehat{\overrightarrow{OI}, \overrightarrow{OI'}})$ and $(\widehat{\overrightarrow{OI'}, \overrightarrow{OI}})$ (See Figure 4.2: right).
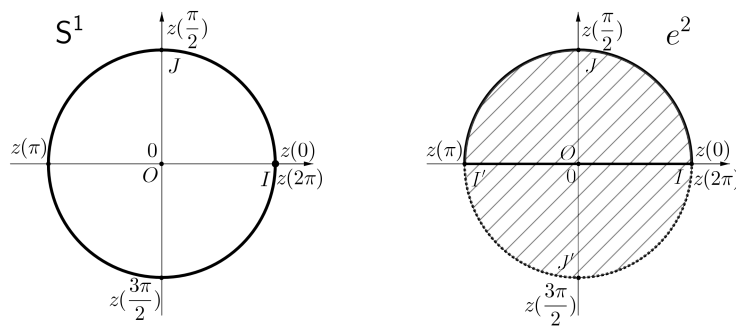

Figure 4.2: S^1 and e^2.

To obtain $X_G = \dfrac{D^2 \amalg S^1}{z \sim z^2}$, we attach the arc $II' \subset S^1$ of the region determined by $(\widehat{\overrightarrow{OI}, \overrightarrow{OI'}})$ to the whole S^1 by the map φ given above with $n = 2$ and the same job is done with the regions determined by $(\widehat{\overrightarrow{OI'}, \overrightarrow{OI}})$.

By Proposition 4.2.2,

$$X_G \cong \frac{D^2}{z \sim z e^{\frac{2\pi i}{2}}}, z \in \partial D^2$$
$$\cong \frac{D^2}{z \sim z e^{\pi i}}, z \in \partial D^2$$
$$\cong \frac{D^2}{z \sim (-z)}, z \in \partial D^2$$
$$\cong \mathbb{R}\mathbf{P}^2.$$

- Case $n = 3$.

  Here, the 2-cell e^2 is considered as divided into three equal parts, regions determined respectively by the oriented angles $(\widehat{\overrightarrow{OI}, \overrightarrow{OA}})$, $(\widehat{\overrightarrow{OA}, \overrightarrow{OB}})$ and determined by $(\widehat{\overrightarrow{OB}, \overrightarrow{OI}})$ (See Figure 4.4: right).
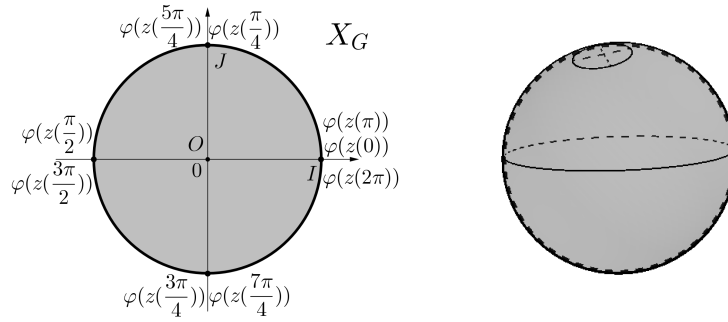
Figure 4.3: $\varphi(\partial D^2)$ and $X_G = \mathbb{R}\mathbf{P}^2$.
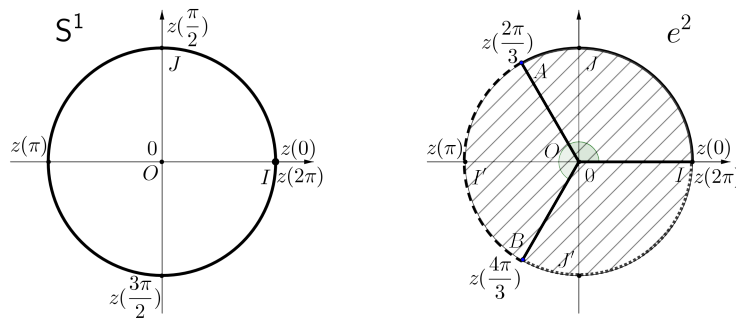

Figure 4.4: S^1 and e^2.

To obtain $X_G = \dfrac{D^2 \amalg S^1}{z \sim z^3}$, we first attach the arc $IA \subset S^1$ of the region determined by $(\overrightarrow{OI}, \overrightarrow{OA})$ to the whole S^1 by the map φ given above with $n = 3$ and the same job is done with the regions determined by $(\overrightarrow{OA}, \overrightarrow{OB})$ and $(\overrightarrow{OB}, \overrightarrow{OI})$ respectively. So we have the following transformations: $\varphi(z(0)) = \varphi(z(\frac{2\pi}{3})) = \varphi(z(\frac{4\pi}{3})) = z(0)$, $\varphi(z(\frac{\pi}{6})) = \varphi(z(\frac{5\pi}{6})) = \varphi(z(\frac{3\pi}{2})) = z(\frac{\pi}{2})$, $\varphi(z(\frac{\pi}{3})) = \varphi(z(\pi)) = \varphi(z(\frac{5\pi}{3})) = z(\pi)$ and $\varphi(z(\frac{\pi}{2})) = \varphi(z(\frac{7\pi}{6})) = \varphi(z(\frac{11\pi}{6})) = z(\frac{3\pi}{2})$ (See Figure 4.3: left). From Proposition 4.2.2, this construction gives us exactly the quotient of the disk D^2 under identification of a point of ∂D^2 to its image by the rotation of angle $\dfrac{2\pi}{3}$, that is, this construction can start by the one of a neighborhood N of the 1-skeleton S^1 of X_G which is the Cartesian product $\mathbb{Y} \times I$ of a graph $\mathbb{Y}$ with the interval I and then we identify the two ends of this product via a one-third twist (See Figures 4.5 and 4.6: right). N is then bounded by one circle, formed by the three endpoints of each $\mathbb{Y}$ cross section of N. The construction of X_G is completed from N by attaching a disk D^2 along the boundary circle of N. This is not a surface and cannot be easily represented since it cannot be done in $\mathbb{R}^3$ [4].

- Case of a larger n.

  Here, the 2-cell e^2 is considered as divided into n equal parts, sectors determined respectively by the oriented angles $(\overrightarrow{OI_k}, \overrightarrow{OI_{k+1}})$, $\forall k = 0, 1, 2, \cdots, n-1$. To obtain $X_G = \dfrac{D^2 \amalg S^1}{z \sim z^n}$, we first attach
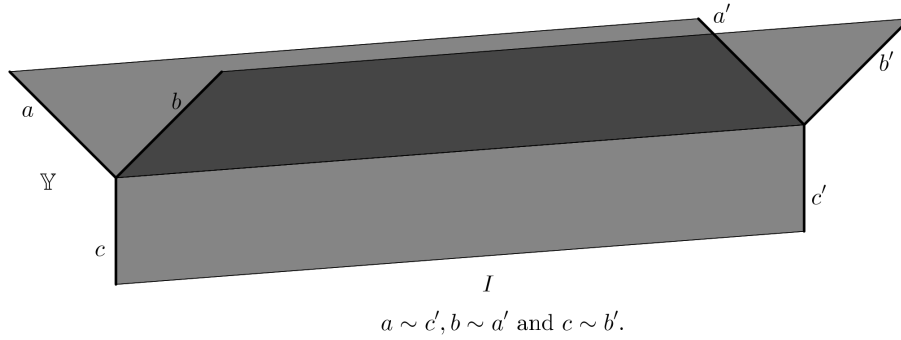
$$a \sim c', b \sim a' \text{ and } c \sim b'.$$

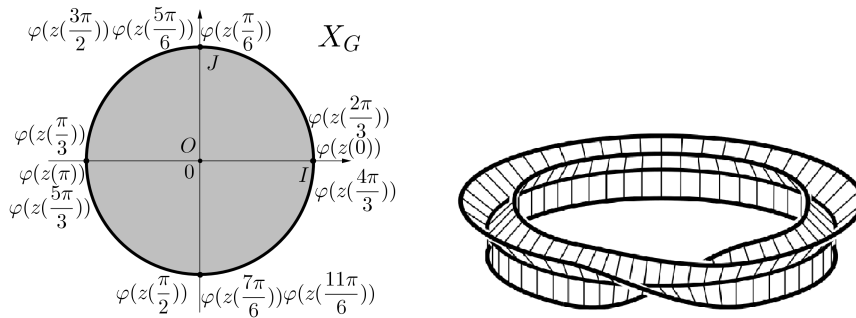Figure 4.5: Construction of N from $\mathbb{Y} \times I$.


Figure 4.6: $\varphi(\partial D^2)$ and the neighborhood N of the 1-skeleton S^1 of X_G [4].

the arc $I_0 I_1 \subset S^1$ of the sector determined by $(\overrightarrow{OI_0}, \overrightarrow{OI_1})$ to the whole S^1 by the map φ given above and the same job is done with the sectors determined respectively by $(\overrightarrow{OI_k}, \overrightarrow{OI_{k+1}}), \forall k = 1, 2, \cdots, n-1$. So we have the following transformations: $\varphi(z(\frac{2k\pi}{n})) = z(0)$, $\varphi(z(\frac{\pi}{2n} + \frac{2k\pi}{n})) = z(\frac{\pi}{2})$, $\varphi(z(\frac{\pi}{n} + \frac{2k\pi}{n})) = z(\pi)$ and $\varphi(z(\frac{3\pi}{2n} + \frac{2k\pi}{n})) = z(\frac{3\pi}{2})$. From Proposition 4.2.2, this construction gives us exactly the quotient of the disk D^2 with identification of a point of ∂D^2 to its image by the rotation of angle $\frac{2\pi}{n}$, that is, this construction can start by the one of a neighborhood N of the 1-skeleton S^1 of X_G, which is the Cartesian product $\mathbb{X} \times I$ of a graph $\mathbb{X}$ with the interval I which is an n-pointed "asterisk" and then we identify the two ends of this product via a $1/n$ twist. N is then bounded by one circle, formed by the n endpoints of each $\mathbb{X}$ cross section of N (See Figure 4.7).

The construction of X_G is completed from N by attaching a disk D^2 along the boundary circle of N. This is not a surface (for $n > 2$) and cannot be easily represented since it cannot be done in $\mathbb{R}^3$.

$$a \sim d', b \sim a', c \sim b' \text{ and } d \sim c'.$$
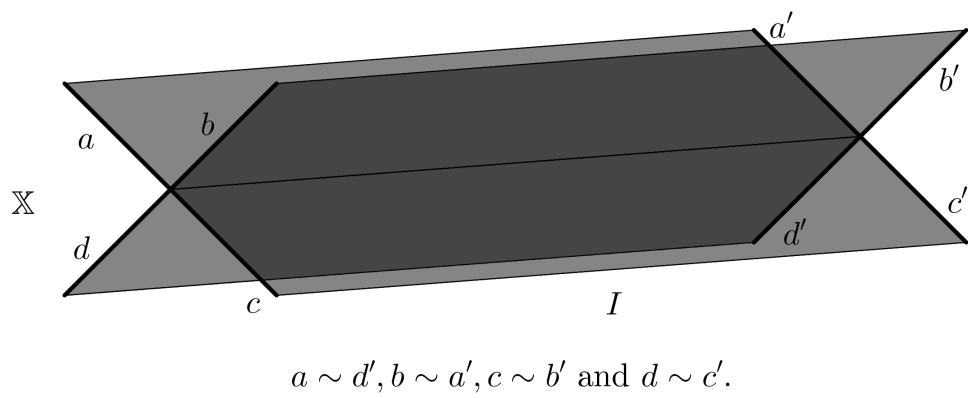
Figure 4.7: Construction of N from $\mathbb{X} \times I$ for $n = 4$.

5. Conclusion

The purpose of this dissertation was to display some applications of fundamental group to cell complexes. We first proved a theorem on how the fundamental group is affected by attaching 2-cells. Then we showed a direct consequence of this theorem saying that the fundamental group of a cell complex X is reduced to the one of its 2-skeleton X^2, that is, $\pi_1(X, x_0) \cong \pi_1(X^2, x_0)$. Finally, we gave a necessary condition for two closed and path-connected surfaces to be homeomorphic and verified the existence of a topological space X_G such that its fundamental group is isomorphic to a given group G. More precisely, we showed that given two closed and path-connected surfaces S_g and S_h both either orientable or nonorientable and of respective genus g and h, a necessary condition for them to be homeomorphic is that their genera must be equal, that is, $g = h$. We also succeeded in constructing a space X_G verifying $\pi(X_G, x_0) \cong G$, where G is a given group. We concluded that considering the presentation $\langle g_\alpha | r_\beta \rangle$ of G, X_G is the 2-dimensional cell complex whose 1-skeleton is the wedge sum $X^1 = \vee_\alpha S^1_\alpha$ generated by the g_α's and the construction of X_G is completed by attaching 2-cells e^2_β to X^1 along the loops associated to the relators r_β. We ended the work with an example of description of the space $X_G = \dfrac{D^2 \amalg S^1}{z \sim z^n}$ $z \in \partial D^2$, where $G = \langle a | a^n \rangle, n \geq 1$. Hence, we have shown that X_G is homeomorphic to $\dfrac{D^2}{z \sim z e^{\frac{2\pi}{n} i}}$ and we concluded that X_G is a surface only for $n = 1, 2$, which are the unit closed disk and the projective plane respectively.

Our work is somehow related to the notion of classification of manifolds. In the future, we plan to go further with that concept.

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