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Mayer-Vietoris sequences

Péguy Kem-Meka Tiotsop Kadzue (kem.peguy@aims-cameroon.org)
African Institute for Mathematical Sciences (AIMS)
Cameroon

Supervised by: Dr. Paul Arnaud Songhafouo Tsopméné
University of Regina, Canada

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Abstract

In this thesis, we explore the Mayer-Vietoris sequence which is one of the key algebraic tools to compute the homology groups of topological spaces. The idea is to decompose a given topological space, say X, into subspaces, say A and B, whose homology groups, including the homology of the intersection $A \cap B$, may be easier to compute. The Mayer-Vietoris theorem claims that if A and B are nice, there is a long exact sequence that relates the homology groups of X to the homology groups of A, B, and $A \cap B$. We provide a detailed proof of the Mayer-Vietoris sequence theorem using some notions in algebraic topology especially some notions in singular homology and the theorem of long exact sequence in homology. Furthermore, we notice that the Mayer-Vietoris sequence theorem can be easily proved using the excision theorem. As applications, we compute the homology of some spaces including the sphere, the wedge of two spaces, the torus, the Klein bottle, and the projective plane. Moreover, we prove some important results in mathematics including the Brouwer fixed point theorem, the invariance of dimension theorem, and we also study the contractibility of the sphere.

Keywords: Exact sequences, singular homology, homotopy invariance.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Péguy KEM-MEKA TIOTSOP KADZUE, 31 May 2019.

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1. Introduction

Algebraic topology can be defined as a branch of mathematics that uses tools from algebra to study topological spaces. Its basic idea is to construct algebraic invariants that classify topological spaces up to homeomorphism, or more precisely up to homotopy equivalence. The well-known invariants, and also the most important ones, include the homotopy groups, homology, and cohomology. This thesis is concerned with homology.

The homology of a topological space X is a sequence of abelian groups $\{H_n(X)\}_{n\geq 0}$, one for each n. The 0th homology group is determined by the number of components of X. For $n \geq 1$, the nth homology group detects "holes of dimension n" (a precise definition is given by Definition 2.2.21). It turns out that it is very hard to compute the homology of most spaces directly from the definition. Fortunately, there is a certain amount of tools that allow to make calculations. One of the most important tools is called Mayer-Vietoris sequence, which we consider in this thesis. One can view the Mayer-Vietoris sequence as the analogue of the well known Seifert-van Kampen theorem [10] for the fundamental group.

1.1 Brief Literature Review

The Mayer-Vietoris sequence is due to two Austrian mathematicians, Walther Mayer and Leopold Vietoris [15]. In fact, Mayer was initiated to topology by his colleague Vietoris when attending his lectures in 1926 at a local university in Vienna [1]. So he was told about the conjectured result of the Mayer-Vietoris sequence and he proved it only for the Betti numbers in 1929 [15]. He applied his results to the torus considered as the union of two cylinders [5]. Vietoris later proved the full result for the homology groups in 1930 but did not express it as an exact sequence [13]. Many years later, in 1952 the concept of an exact sequence appeared in the book Foundations of Algebraic Topology [6] by Samuel Eilenberg and Norman Steenrod, where for the first time the tool Mayer-Vietoris sequence was expressed in the modern form.

Nowadays, the Mayer-Vietoris sequence is very helpful in Topological Data Analysis (TDA) [17]. Actually, TDA is a new area of research that uses algebraic topology to extract non-linear features from data sets. It is based on the computation of the so-called persistent homology [2], which measure the features of shapes and functions of big data sets. So there exists an analogue of Mayer-Vietoris sequence in TDA for computing persistent homology [4]. For instance, when the data are robotics network, looking at them as a sequence of spaces, we can apply the Mayer-Vietoris sequence to relate the homology groups of the space to the homology groups of its subspaces. So by the mathematical definition of a robotic network in [14], we can split the set of robots to apply the Mayer-Vietoris sequence, choose a couple of subspaces A, B of X, where X is the union of the interiors of A and B, the Mayer-Vietoris sequence has the form:

$$\cdots \to H_n(A\cap B) \xrightarrow{\phi} H_n(A)\oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \to \cdots \to H_0(X) \to 0.$$

1.2 Objectives of the Study

The objectives of this work are the following:

1. To prove the Mayer-Vietoris sequence theorem (Theorem 3.3.2).
2. To compute using the Mayer-Vietoris sequence the homology of some spaces.
3. To go over some general well known results in mathematics, including the Brouwer fixed-point theorem and the invariant of dimension theorem. And also the problem of the non-contractibility of the sphere.

1.3 Outline of the Thesis

This essay is divided into five chapters. In the first chapter, we introduce the topic, briefly review the literature, and then state the objectives. In Chapter 2, after reviewing the general concepts of homotopy theory, we establish the basics of singular homology which are the ingredients for proving the Mayer-Vietoris sequence theorem. In Chapter 3, we present the proof with all the details of the latter theorem. In Chapter 4, we present some applications. Finally, in Chapter 5, we give the conclusion of the work.

2. Preliminaries

To achieve the objectives we have set, we first have to describe the tools needed. So in this chapter, we first present notions of homotopy, and then alternatively we define the notion singular of homology, exact sequences, relative homology and reduced homology.

Throughout this work, X and Y are usually two topological spaces, and by map between X and Y we mean a continuous map.

2.1 Homotopy Equivalence

2.1.1 Definition (Homotopy). [10] Let $f_0, f_1 : X \to Y$ be two maps. A homotopy between $f_0, f_1 : X \to Y$ is a map

$$\begin{aligned} F : [0,1] \times X &\longrightarrow Y \\ (x,t) &\longmapsto F(x,t) \end{aligned}$$

such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$. If such an F exists, we say f_0 is homotopic to f_1, and write $f_0 \backsimeq f_1$.

2.1.2 Proposition. [10] The relation $\simeq$ is an equivalence relation on the set of maps from X to Y.

2.1.3 Definition (Homotopy equivalence). [10] A map $f : X \to Y$ is a homotopy equivalence if there is another map $g : Y \to X$ such that $f \circ g \simeq \mathbb{1}_Y$ and $g \circ f \simeq \mathbb{1}_X$. We say that X is homotopy equivalent to Y or X and Y have the same homotopy type, and we write $X \simeq Y$.

2.1.4 Definition (Deformation retraction). [10] Let A be a subspace of X. We say that A is a retract of X if there is a continuous map $r : X \to A$ such that $r \circ i = \mathbb{1}_A$. If in addition, $i \circ r \simeq \mathbb{1}_X$, we say that A is a deformation retract of X.

The following proposition gives a useful deformation retract.

2.1.5 Proposition. For $n \geq 0$, $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ is a deformation retract of $\mathbb{R}^{n+1} - \{0\}$.

Proof. Let $i : S^n \to \mathbb{R}^{n+1} - \{0\}$ be the inclusion. Let us consider $r : \mathbb{R}^{n+1} - \{0\} \to S^n$ defined by

$$r(x) = \frac{x}{||x||}.$$

So r is continuous as a quotient of continuous function. Since $y \in S^n$ it follows that $||y|| = 1$, and we have $r \circ i = \mathbb{1}_{S^n}$ so $r \circ i$ is the identity. Let us show now that $i \circ r$ is homotopic to the identity. We consider the map

$$F(x,t) = tx + (1-t)r(x).$$

So $F(x,t) = x\big[t + (1-t)\frac{1}{||x||}\big]$, F never vanishes. Hence F is well defined. Moreover F is continuous because r is. In addition, $F(x,0) = r(x)$ and $F(x,1) = x$. So S^n is a deformation retract of $\mathbb{R}^{n+1} - \{0\}$.

□

2.1.6 Definition (Contractible spaces). [10] A topological space X is said to be contractible if $X \backsimeq \{*\}$.

2.1.7 Example. For all, $n \geq 0$, the space $\mathbb{R}^n$ and the disk D^n are contractible.

2.2 Singular Homology

2.2.1 Chain complexes.

2.2.2 Definition (Chain complex). [10] A chain complex is a collection $\{A_n\}_{n\in\mathbb{Z}}$, denoted A_*, of abelian groups equipped with group homomorphisms $\partial_n : A_n \to A_{n-1}$ such that

$$\partial_n \circ \partial_{n+1} = 0 \quad \text{for all} \quad n \in \mathbb{Z}. \tag{2.2.1}$$

2.2.3 Definition. Let A_* be a chain complex. An element a of A_n is called

(i) an n-cycle (or just a cycle) if $\partial_n(a) = 0$;

(ii) an n-boundary (or just a boundary) if there exists $b \in A_{n+1}$ such that $\partial_{n+1}(b) = a$.

2.2.4 Remark. The equation 2.2.1 implies that im∂_{n+1} is a subgroup of Ker∂_n.

2.2.5 Definition (Homology). [10] The homology of a chain complex (A_*, ∂_*) is the graded group $H_*(A) := \text{Ker}\partial_*/\text{Im}\partial_{*+1}$. In other words,

$$H_n(A) = \text{Ker}\partial_n/\text{Im}\partial_{n+1}$$

for all $n \in \mathbb{Z}$.

2.2.6 Example. Let us consider the chain complex described by the following sequence :

$$\cdots \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \cdots .$$

Their homologies groups are alternately isomorphic to $\frac{\mathbb{Z}}{2\mathbb{Z}}$ and $\{0\}$.

2.2.7 Definition (Chain map). [10] Let A_* and B_* be chain complexes. A chain map from $f : A_* \longrightarrow B_*$ is a collection $\{f_n : A_n \longrightarrow B_n\}_{n\in\mathbb{Z}}$ of group homomorphisms such that

$$\partial_n f_n = f_{n-1}\partial_n \quad \text{for all} \quad n \in \mathbb{Z}. \tag{2.2.2}$$

This amounts to saying that the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial_{n+1}} & A_n & \xrightarrow{\partial_n} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow[\partial_{n+1}]{} & B_n & \xrightarrow[\partial_n]{} & B_{n-1} & \longrightarrow & \cdots .
\end{array}$$

2.2.8 Definition (Chain homotopy). [10] Let A_*, B_* be chain complexes, and $f_0, f_1 : A_* \to B_*$ be chain maps. A chain homotopy h between f_0 and f_1 is a collection of homomorphisms $h : A_n \to B_{n+1}$ such that

$$\partial h + h\partial = f_1 - f_0.$$

2.2.9 Simplices.

2.2.10 Definition (The n-simplex). [10] Consider a Euclidean space $\mathbb{R}^m, m \geq 0$ and let $n \leq m$. Let $v_0, \cdots, v_n$ be a sequence of points of $\mathbb{R}^m$ such that the difference vectors $v_1 - v_0, \cdots, v_n - v_0$ are linearly independent. Define $[v_0, \cdots, v_n]$ as

$$[v_0, \cdots, v_n] = \left\{ \sum_{i=0}^{n} t_i v_i : \quad (\forall i : t_i \geq 0) \quad \text{and} \quad \sum_{i=1}^{n} t_i = 1 \right\}.$$

This set is called the n-simplex generated by $v_0, \cdots, v_n$. The coefficients t_i are called the barycentric coordinates of the point $\sum_{i=0}^{n} t_i v_i$ in $[v_0, \cdots, v_n]$. The points $v_0, \cdots, v_n$ are called the vertices of $[v_0, \cdots, v_n]$.

2.2.11 Definition (Standard n-simplex). [3] For $n \geq 0$, The standard n-simplex is of points is the subset of $\mathbb{R}^{n+1}$ given by $\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}$.

2.2.12 Example. [10] The $n+1$ vertices of the standard n-simplex are the points $e_i \in \mathbb{R}^{n+1}$, where

$$e_0 = (1, 0, 0, \cdots, 0), e_1 = (0, 1, 0, \cdots, 0), \cdots, e_n = (0, 0, 0, \cdots, 1).$$

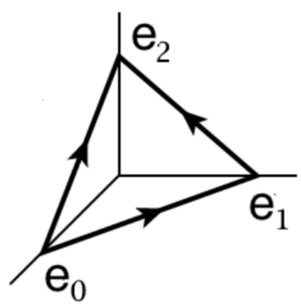


Figure 2.1: Standard 2-simplex [3].

2.2.13 Definition. [10] Let $[v_0, \cdots, v_n]$ be an n-simplex, and let $0 \leq k \leq n$. The $(n-1)$-simplex generated by $v_0, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n$ is called a face of $[v_0, \cdots, v_n]$, and it is denoted $[v_0, \cdots, \hat{v_k}, \cdots, v_n]$.

Now we want to define the notion of singular homology of a given topological space X. We will see that from any topological space X, we can get the set of singular n-simplices. So we will define a singular chain complex in terms of finite formal sums.

2.2.14 Definition (Singular n-simplex). [10] Let X be a space. A singular n-simplex in X is a continuous function (also called a map) $\sigma : \Delta^n \longrightarrow X$.

2.2.15 Example. [10] The inclusion of the standard n-simplex into $\mathbb{R}^{n+1}$ is a singular n-simplex.

2.2.16 Singular homology.

2.2.17 Definition ($C_n(X)$)**.** [10] Given a topological space X and $n \geq 0$. We define $C_n(X)$ to be the free abelian group generated by all singular n-simplices in X. An element of $C_n(X)$ is called a singular n-chain of X.

Moreover elements of $C_n(X)$ are called n chains, or more precisely singular n chains, and are finite formal sums $\sum_i n_i\sigma_i$ for $n_i \in \mathbb{Z}, \quad \sigma_i : \Delta^n \longrightarrow X \quad i \in I$.

2.2.18 Definition (Boundary map)**.** [10] The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is the homomorphism where $\partial_n(\sigma) = \sum_{i=0}^{n}(-1)^i\sigma|[v_0, \dots, \hat{v}_i, \dots, v_n]$.

2.2.19 Proposition. For all $n \geq 0, \quad \partial_{n-1} \circ \partial_n = 0$.

Proof. We have : $\partial_n(\sigma) = \sum_{i=0}^{n}(-1)^i\sigma|[v_0, \dots, \hat{v}_i, \dots, v_n]$ and hence

$$\begin{aligned}\partial_{n-1}\partial_n(\sigma) &= \sum_{j<i}(-1)^{i+j}\sigma|[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \cdots, v_n] \\ &- \sum_{j>i}(-1)^{i+j}\sigma|[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \cdots, v_n] \\ &= 0.\end{aligned}$$

In fact, the two summations cancel by switching i and j in the second sum. ☐

2.2.20 Corollary (Singular chain complex)**.** Given a topological space X, $(C_*(X), \partial_*)$ is a chain complex.

Proof. The preceding Proposition 2.2.19 gives that $\partial \circ \partial = 0$. ☐

2.2.21 Definition (Singular homology group)**.** [10] For all $n \geq 0$, the singular homology of X is defined to be the homology of the chain complex $(C_n(X), \partial_n)$. That is,

$$H_n(X) = \mathrm{ker}\partial_n / \mathrm{Im}\partial_{n+1}.$$

The group $H_n(X)$ is called the nth singular homology group of X.

2.2.22 Example. [10] If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$. In fact here is a unique singular n-simplex $\sigma_n : \Delta^n \longrightarrow \{*\}$ for each n, so $C_n(\{*\}) = \mathbb{Z}$ and $\partial(\sigma_n) = \sum_{i=0}^{n}(-1)^i\sigma_{n-1}$, a sum of $n+1$ terms, which is therefore 0 for n odd and σ_{n-1} for n even, $n \neq 0$. Thus we have the chain complex:

$$\cdots \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0,$$

the homology groups of this complex are trivial except for $H_0(X) \cong \mathbb{Z}$.

We will move on to homotopy invariance of singular homology.

2.2.23 Homotopy invariance.

2.2.24 Definition (Induced map)**.** [10] Let $f : X \to Y$ be a map between two topological spaces. The induced map $f_\sharp : C_n(X) \to C_n(Y)$ is defined by composing each singular n-simplex $\sigma : \Delta^n \to X$ with f to get a singular n-simplex $f_\sharp(\sigma) = f \circ \sigma$, and extending $f_\sharp$ by linearity. We have the following commutative diagram:

$$\begin{array}{ccc} & \overset{\sigma}{\nearrow} & X \\ \Delta^n & & \Big\downarrow f \\ & \underset{f_\sharp(\sigma)}{\searrow} & Y \end{array}$$

2.2.25 Proposition. Let $f : X \to Y$ be a map between two topological spaces. The induced homomorphism $f_\sharp : C_n(X) \to C_n(Y)$ is a chain map.

Proof. Let us consider the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$. Then we have :

$$\begin{aligned} f_\sharp \partial_n(\sigma) &= f_\sharp\Big(\sum_{i=0}^{n} (-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n]\Big) \\ &= \sum_{i=0}^{n} (-1)^i (f \circ \sigma)|[v_0, \ldots, \hat{v}_i, \ldots, v_n] \\ &= \partial_n f_\sharp(\sigma), \quad \text{i.e.} \quad f_\sharp \partial(\sigma) = \partial f_\sharp(\sigma). \end{aligned}$$

Thus we have a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow & \cdots \\ & & \Big\downarrow f_\sharp & & \Big\downarrow f_\sharp & & \Big\downarrow f_\sharp & & \\ \cdots & \longrightarrow & C_{n+1}(Y) & \xrightarrow[\partial_{n+1}]{} & C_n(Y) & \xrightarrow[\partial_n]{} & C_{n-1}(Y) & \longrightarrow & \cdots . \end{array}$$

So $f_\sharp$ is a chain map from the singular chain complex of X to that of Y.

□

2.2.26 Proposition. Let $f : X \to Y$ be a map between two topological spaces. Then $f_\sharp$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.

Proof. By Proposition 2.2.25, the induced homomorphism $f_\sharp : C_n(X) \to C_n(Y)$ is a chain map from the singular chain complex of X to that of Y. So we have the relation $f_\sharp(\sigma) = f \circ \sigma$, implies that $f_\sharp$ takes cycles to cycles since $\partial\alpha = 0$ implies $\partial(f_\sharp\alpha) = f_\sharp(\partial\alpha) = 0$. Also, $f_\sharp$ takes boundaries to boundaries since $\partial(f_\sharp\beta) = f_\sharp(\partial\beta)$. Hence the proposition holds. □

2.2.27 Definition. The homomorphism $f_*: H_n(X) \to H_n(Y)$ is called the induced homomorphism by f in homology.

2.2.28 Proposition. Let $f: X \to Y$ and $g: Y \to Z$ be maps. Then

(i) $(g \circ f)_* = g_* \circ f_*$,

(ii) $(1_X)_* = \mathbb{1}_{H_*(X)}$.

Proof. This follows immediately from the definition. □

We will now state some powerful results in singular homology.

2.2.29 Theorem (Homotopy invariance theorem)**.** *[10] If two maps $f, g : X \to Y$ are homotopic, then they induce the same maps on homology, i.e. $f_* = g_* : H_n(X) \to H_n(Y)$.*

From that homotopy invariance Theorem 2.2.29, we have the following corollary.

2.2.30 Corollary. [10] If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism. In other words if X is homotopy equivalent to Y, then $H_*(X)$ is isomorphic to $H_*(Y)$.

Proof. Let suppose that f is a homotopy equivalence, if $g : Y \to X$ is a homotopy inverse, then

$$g_* \circ f_* = (g \circ f)_* = (\mathbb{1}_X)_* = \mathbb{1}_{H_n(X)}.$$

On the other hand :

$$f_* \circ g_* = (f \circ g)_* = (\mathbb{1}_Y)_* = \mathbb{1}_{H_n(Y)}.$$

So f_* is an isomorphism with an inverse g_*. □

2.3 Exact Sequences

Here we are interested in the notions of exact sequences.

2.3.1 Definition. [10] A triple $A \xrightarrow{f} B \xrightarrow{g} C$ of abelian groups and homomorphisms is exact if Im$(f) =$ ker(g). A sequence of abelian groups and homomorphisms

$$\cdots \to M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} \cdots$$

is exact if each triple $M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2}$ is exact for all i.

2.3.2 Definition. [10] An exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called short exact.

2.3.3 Proposition. [10] Suppose we have the exact sequences

$$0 \to A \xrightarrow{f} B$$

and

$$B \xrightarrow{g} C \to 0.$$

If and only if f is injective and g is surjective.

Proof. The homomorphism $0 \to A$ is just the trivial homomorphism with image 0, so $\mathrm{Ker}(f) = 0$ which means that f is injective. The homomorphism $C \to 0$ maps all of C to 0 and hence has kernel C. Since the sequence is exact we have $\mathrm{Im}(g) = C$ and g is surjective; the other implication is straightforward. □

2.3.4 Corollary. [10] Given the following short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

f is injective and g is surjective. If

$$0 \to A \xrightarrow{f} B \to 0$$

is exact, then f is an isomorphism.

Proof. Straightforward from the preceding, Proposition 2.3.3. □

All those three following lemmas are there to prepare the proof of the great result which is: the short exact sequences of complexes give rise to long exact sequences in homology (Theorem 2.3.8).

2.3.5 Lemma (Existence of two homomorphisms)**.** Let

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$$

be a short exact sequence of chain complexes. Then for all n there exists two homomorphisms

$$i_* : H_n(A_\bullet) \to H_n(B_\bullet) \quad \text{and} \quad j_* : H_n(B_\bullet) \to H_n(C_\bullet).$$

Proof. With a slight abuse of notation we will refer to the graded homomorphisms of all of the chain complexes as ∂, so have the chain complexes $(A_\bullet, \partial), (B_\bullet, \partial), (C_\bullet, \partial)$, and using the fact that

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$$

is a short exact sequence of chain complexes.

We get the following commutative diagram by letting n vary :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow i & & \downarrow i & & \downarrow i & & \\
\cdots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow j & & \downarrow j & & \downarrow j & & \\
\cdots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which the columns are exact and the rows are chain complexes. The commutativity of the squares in the short exact sequence of chain complexes implies that i and j are chain maps: $i \circ \partial = \partial \circ i$ and $j \circ \partial = \partial \circ j$. The image of any boundary is a boundary, and the image of any cycle is a cycle. They induce homomorphisms $i_* : H_n(A_\bullet) \to H_n(B_\bullet)$ and $j_* : H_n(B_\bullet) \to H_n(C_\bullet)$. □

2.3.6 Lemma (Connecting Homomorphism). [10] Let

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$$

be a short exact sequence of chain complexes. Then for all n there exists a canonical homomorphism

$$\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet).$$

called connecting homomorphism which is often also called boundary operator.

Proof. We want to prove the existence of the ∂. To do so we must construct the map $\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$, show that it is well defined and then show that it is a homomorphism.

- Construction of ∂. Let $c \in C_n$ be a cycle. Using the fact that j is into, we can write $c = j(b)$ for some $b \in B_n$. We also have $\partial b \in B_{n-1}$ in Ker$j$, because $j(\partial b) = \partial j(b) = \partial c = 0$. That implies $\partial b = i(a)$ for some $a \in A_{n-1}$ because Ker$j$ = Imi. We also have $\partial a = 0$ since $i(\partial a) = \partial i(a) = \partial\partial b = 0$ and i is injective. Then we can define $\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ by sending the homology class of c to the homology class of a, so $\partial[c] = [a]$.

- ∂ is well defined. In fact, a is uniquely determined by ∂b since i is injective. Moreover, if we pick a different value b' instead of b where $j(b') = j(b) = c$, we will get $j(b'-b) = 0$ so $b'-b \in$ ker$j$ = Imi. $b' - b = i(a')$ for some a', so $b' = b + i(a')$. Replacing by $a + \partial a'$ we get

$$i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial i(a') = \partial(b + i(a')) = \partial b'.$$

  Indeed the element $a + \partial a'$ has the same relation with a, so we get the same homology class for $\partial[c]$.

In the similar way, if we pick a different value c' in the homology class of c, we replace c with $c' = c + \partial c'$ for some value $c'' \in C_{n+1}$. $c = j(b'')$ for some $b'' \in B_{n+1}$.

$$c' = c + \partial j(b'') = c + j(\partial b'') = j(b + \partial b'').$$

So b is replaced by $b + \partial b''$, which leaves ∂b unchanged and then we get the same value for a. we conclude that the homology class does not depend on the picked element. Since we get the same homology class for $\partial[c]$. This is to say that our map ∂ is well defined.

- ∂ is a homomorphism. actually if $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a_2]$ via elements b_1 and b_2 as above. $j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$. Thus computing $\partial([c_1] + [c_2])$ as above we get $[a_1] + [a_2]$, as wished.

  So ∂ is a homomorphism.

□

2.3.7 Remark. The key notion of the above Lemma is that : the connecting homomorphism $\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ is the map sending the homology class of c to the homology class of a i.e. $\partial[c] = [a]$.

2.3.8 Theorem (The homology long exact sequence). *[10] A short exact sequence of chain complexes* $0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$ *induces a long exact sequence in homology groups*

$$\cdots \to H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \xrightarrow{i_*} H_{n-1}(B_\bullet) \to \cdots . \tag{2.3.1}$$

Proof. From the Lemma 2.3.5 and Lemma 2.3.6, we already have the existence of i_*, j_* and ∂ so it remains just to show that the sequence 5.0.1 is exact. To do so we will proceed in three steps.

Step 1, The goal of this step is to prove that Imi_* = Kerj_*. So we start by checking Im$i_* \subset$ Kerj_*, this is straightforward, because $j \circ i = 0$ in the short exact sequence and this implies that, now using that fact that H_n is a covariant functor we get $j_* \circ i_* = 0$.
Now let check Ker$j_* \subset$ Imi_*, we consider a representative cycle $b \in B_n$ for a homology class in the kernel of j_*. $j_*([b]) = 0$, so $j(b) = \partial c'$ for some $c' \in C_{n+1}$. Since j is surjective at each dimension, $c' = j(b')$ for some $b' \in B_{n+1}$. We evaluate $j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial j(b') = j(b) - \partial c' = j(b) - j(b) = 0$. $(b - \partial b') \in$ ker$j$ = Imi. Thus $b - \partial b' = i(a)$ for some $a \in An$. $i(\partial a) = \partial i(a) = \partial(b - \partial(b')) = \partial b = 0$ since b is a cycle. By injectivity of i, then, $\partial a = 0$ and a is a cycle with a homology class $[a]$. $i_*([a]) = [b - \partial b'] = [b]$, so $[b] \in$ Imi_*.

Step 2, Here we want to prove Imj_* = Ker∂. If $[c] \in$ Imj_*, then b as defined when calculating $\partial[c]$ has a homology class and is therefore a cycle. $\partial b = 0$ so $\partial([c]) = [a] = 0$. Thus $[c] \in$ Ker∂.
Now assume $[c]$ is in Ker∂. $\partial[c] = [a] = 0$ so $a = \partial a'$ for some $a' \in A_n$. $\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = \partial b - \partial b = 0$, so the element $(b - i(a'))$ is a cycle in B_n and has a homology class $[b - i(a_0)]$. $j(b - i(a')) = j(b) - ji(a_0) = j(b) = c$ so $j_*([b - i(a_0)]) = [c]$ and thus $[c] \in$ Imj_*.

Step 3, It remain to check that Im∂ = Keri_*. i_* takes $\partial[c] = [a]$ to $[\partial b]$, which is 0, so Im$\partial \subset$ Keri_*.
A homology class in Keri_* is represented by an cycle $a \in A_{n-1}$ where $i(a) = \partial b$ for some $b \in B_n$. $\partial j(b) = j(\partial b) = ji(a) = 0$, so $j(b)$ is a cycle and has a homology class $[j(b)]$. The homomorphism ∂ takes $[j(b)]$ to $[a]$, and thus $[a] \in$ Im∂.

□

2.4 Reduced and Relative Homology

It is often very useful to have a modified version of homology for which a point has trivial homology groups in all dimensions, generally we want to get something like $\widetilde{H}_0(X) = 0$ for a point, where $\widetilde{H}$ stands for the reduced homology that we now define.

2.4.1 Reduced Homology.

2.4.2 Definition. [10] Let X be a non-empty topological space, The reduced homology $\widetilde{H}$ is the homology of the augmented chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where the homomorphism $C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$ is defined by $\varepsilon(\sum_i n_i\sigma_i) = \sum_i n_i$, so $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$ and $H_n(X) \cong \widetilde{H}_n(X)$ for all $n > 0$.

2.4.3 Remark. The relations $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$ and $H_n(X) \cong \widetilde{H}_n(X)$ for all $n > 0$, are meaningful. In fact, $\varepsilon \circ \partial_1 = 0$, so ε vanishes on Im∂_1 and hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\tilde{H}_0(X)$. Obviously $H_n(X) \cong \widetilde{H}_n(X)$ for all $n > 0$.

2.4.4 Corollary. If X is a non-empty contractible space then $\widetilde{H}_n(X) = 0$ for all $n \geq 0$.

Proof. We have X contractible that implies that X is homotopy equivalent to the point $\{*\}$. By using Proposition 2.2.30, we get $\widetilde{H}_n(X) \cong \widetilde{H}_n(\{*\})$, but $H_0(X) \cong \widetilde{H}_n(X) \oplus \mathbb{Z} \cong \mathbb{Z}$ by definition. Hence $\widetilde{H}_n(X) \cong \widetilde{H}_n(\{*\}) = 0$. □

Let us move on to relative homology. In fact, relative homology is seen as a generalization of a homology theory on topological spaces X to a homology theory on pairs of spaces (X, A), where $A \subset X$.

2.4.5 Relative Homology.

2.4.6 Proposition. [10] Given a space X and a subspace $A \subset X$, let $C_n(X, A) = C_n(X)/C_n(A)$ be the quotient group, the boundary map $\partial : C_n(X) \to C_{n-1}(X)$ induces a quotient boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$ so that when n varies we have a sequence of boundary maps
$\cdots \to C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \cdots$ i.e. $\partial \circ \partial = 0$.

2.4.7 Definition (Relative Chain Complex)**.** Let $A \subset X$, be a subspace of a topological space X. The relative chain complex of the pair (X, A) is $C_\bullet(X, A) = C_\bullet(X)/C_\bullet(A)$. And for $n \geq 0$, We call $C_n(X, A) = C_n(X)/C_n(A)$ the nth relative chain complex of X and A.

2.4.8 Remark. This definition makes sense, since we have from Proposition 2.4.6, a sequence
$\cdots \to C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A)) \xrightarrow{\partial} C_{n-2}(X, A) \to \cdots$ with $\partial \circ \partial = 0$.

2.4.9 Definition (Relative Homology)**.** Let $A \subset X$, be a subspace of a topological space X. The relative homology of the pair (X, A) is the homology of the relative chain complex $C_\bullet(X, A)$, i.e. $H_n(X, A) = H_n(C_\bullet(X, A))$.

We are now going to give some corollaries showing the relationship between relative Homology and reduced Homology. Actually, the following is a consequence of Theorem 2.3.8.

2.4.10 Corollary (Reduced Long Exact Homology Sequence)**.** Let (X, A) be a pair with $A \neq \emptyset$, we get a long exact sequence for the reduced homology of relative homology:

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X, A) \to \tilde{H}_{n-1}(A) \to \tilde{H}_{n-1}(X) \to \cdots \to H_0(X, A) \to 0.$$

2.4.11 Example. For all $n > 0$, let us consider the disks $D^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \leq 1\}$. We have

$$H_i(D^n, \partial D^n) \cong H_{i-1}(S^{n-1}) \quad \text{for all} \quad i > 0.$$

Where $\partial D^n = S^{n-1}$ is the unit sphere. In fact, Applying the exact sequence of reduced homology to the pair $(D^n, \partial D^n)$ we get the sequence:

$$\cdots \to \tilde{H}_n(D^n) \to H_n(D^n, S^{n-1}) \to \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(D^n) \to \cdots .$$

Because the disk is contractible, we know its reduced homology groups vanish in all dimensions, so the above sequence collapses to the short exact sequence:

$$0 \to H_i(D^n, S^{n-1}) \to \tilde{H}_{i-1}(S^{n-1}) \to 0 \quad \text{for all} \quad i > 0.$$

So $H_i(D^n, S^{n-1}) \cong \tilde{H}_{i-1}(S^{n-1})$ for all $i > 0$.

2.4.12 Remark. We also have the long exact sequence for relative homology given in the similar way as in the preceding corollary (Corollary 2.4.10).

2.4.13 Corollary. [10] Let X be a space, $x_0 \in X$ be a basepoint. And let $A \subset X$, be a non-empty subspace of X, we have the following results:

(i) $\widetilde{H}_n(\{x_0\}) = 0$ for all n,

(ii) $\widetilde{H}_n(X) = H_n(X, \{x_0\})$ for all n,

(iii) $\widetilde{H}_n(X, A) = H_n(X, A)$ for $n \geq 1$.

3. Mayer-Vietoris Sequences

In this chapter we will discuss the important Mayer-Vietoris sequence, which is one of the key tools for computing the singular homology of various topological spaces. One can think of the Mayer-Vietoris sequence as a way to "glue" the homology of subspaces, easier to calculate, in order to get the homology of the big space. It turns out that the proof of this involves the notion of barycentric subdivision.

3.1 The Subdivision Chain Map

3.1.1 Definition (Barycenter)**.** The barycenter of an n-simplex $[v_0, ..., v_n]$ is the point b of the form $b = \sum_i t_i v_i$, where $t_i = 1/(n+1)$ for all i.

3.1.2 Definition (Barycentric subdivision)**.** Let $n \in \mathbb{N}$, the barycentric subdivision of an n-simplex $[v_0, \cdots, v_n]$ is defined by induction on n. For $n = 0$, the barycentric subdivision of $[v_0]$ is defined to be $[v_0]$ itself. Assume that the barycentric subdivision is defined for $(n-1)$-simplices. Define the barycentric subdivision of $[v_0, \cdots, v_n]$ as the decomposition of $[v_0, \cdots, v_n]$ into the n- simplices $[b, w_0, \cdots, w_{n-1}]$, where b is the barycenter of $[v_0, \cdots, v_n]$, and $[w_0, \cdots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ for $0 \leq i \leq n$.

3.1.3 Example. The cases $n = 1, 2$ and part of the case $n = 3$ are shown in the figure below.

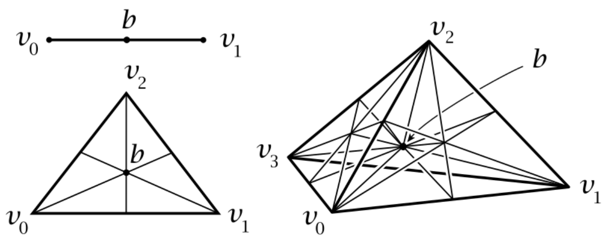


Figure 3.1: Barycentric subdivision [10].

3.1.4 Definition (Diameter of a simplex)**.** The diameter of a simplex $[v_0, \cdots, v_n]$ is the maximum distance between any two of its points, it is denoted by $diam([v_0, \cdots, v_n])$.

3.1.5 Lemma. The diameter of every simplex 3.1.4 in the barycentric subdivision 3.1.2 of $[v_0, \cdots, v_n]$ is bounded from above by $n/(n+1)$ times the diameter of $[v_0, \cdots, v_n]$ i.e. if Δ' is a simplex in the barycenter subdivision of Δ then $diam(\Delta') \leq \frac{n}{n+1} diam(\Delta)$.

Proof. We prove this by induction on n.

The base case, when $n = 0$ holds, because $\Delta^0 = [v_0]$ has just one simplex and its barycentric subdivision is itself.

Let us assume that this is true for $n-1$. Let $b = \sum_i t_i v_i$, with $\sum_i t_i = 1$ be the barycenter of $\Delta = [v_0, \cdots, v_n]$.

Then the maximum distance between two points v and b of Δ satisfies the inequality

$$|v - \sum_i t_i v_i| = |\sum_i t_i(v - v_i)| \leq \sum_i t_i |v - v_i| \leq \sum_i t_i max|v - v_i| = max|v - v_i|.$$

Indeed we need to verify that the distance between any two vertices w_j and w_k of a simplex $[w_0, \cdots, w_n]$ of the barycentric subdivision of $[v_0, \cdots, v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0, \cdots, v_n]$. Then we have two cases:

If b is not a vertex of $\Delta^{'}$ then $\Delta^{'}$ lies in $\partial\Delta$ and so we are done by induction.

Suppose b is a vertex of $\Delta^{'}$. Then; $\Delta^{'} = [b, w_0, \cdots, w_{n-1}]$, where $[w_0, \cdots, w_{n-1}]$ is a simplex in the barycentric subdivision of the face, $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ of Δ.

Let b_i be the barycenter of $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$. Then the line through v_i and b meets $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ in the barycenter b_i of $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$.

Then we have $b = \frac{1}{n+1} v_i + \frac{n}{n+1} b_i$ because b_i be the barycenter of $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$.

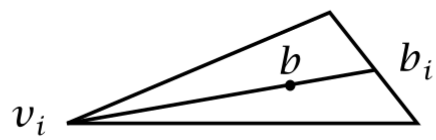


Figure 3.2: Barycenter [10].

So we have :

$$|b - v_i| = \frac{n}{n+1}|v_i - b_i| \leq \frac{n}{n+1} diam(\Delta).$$

And so it follows by the induction hypothesis that:

$$diam(\Delta^{'}) = max_{i \neq j}|w_i - w_j| \leq \frac{n-1}{n} diam(\Delta) \leq \frac{n}{n+1} diam(\Delta)$$

where w_i, w_j are vertices of Δ. □

In the sequel, Y is a convex subset in some $\mathbb{R}^m, m \geq 0$.

3.1.6 Definition (Linear simplex). Let $\Delta^n = [v_0, \cdots, v_n]$, with $n \leq m$. A singular simplex $\lambda : \Delta^n \to Y$ is called linear if for every element $x = \sum_i t_i v_i$ in Δ^n, we have $\lambda(x) = \sum_i t_i \lambda(v_i)$.

3.1.7 Definition (Linear chain). Let $\Delta^n = [v_0, \cdots, v_n]$, with $n \leq m$. A linear chain of Y (more precisely linear n-chain) is a formal finite sum $\sum_i n_i \lambda_i$, where every λ_i is a linear simplex of Y.

3.1.8 Definition ($LC_\bullet(Y)$). We define $LC_\bullet(Y)$ to be the subgroup of $C_\bullet(Y)$ whose elements are linear chains.

3.1.9 Remark. $(CL_\bullet(Y), \partial)$ is a chain complex where the boundary map $\partial : C_\bullet(Y) \longrightarrow C_{\bullet-1}(Y)$ takes $LC_n(Y)$ to $LC_{n-1}(Y)$.

3.1.10 Definition (Cone operator b)**.** Let $b \in Y$. Then the homomorphism $b : LC_n(Y) \to LC_{n+1}(Y)$ defined by $b([w_0, \cdots, w_n]) = [b, w_0, \cdots, w_n]$ is called cone operator.

3.1.11 Example (Example of representation for $n = 2$)**.** Here is an example for $n = 2$, so we have $\Delta^2 = [w_0, w_1, w_2]$.

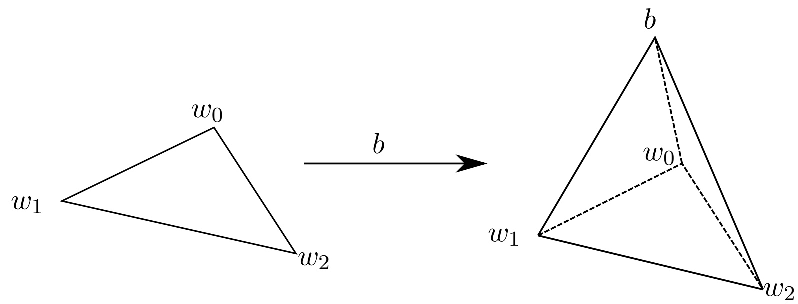


Figure 3.3: Example of representation (cone operator).

3.1.12 Proposition. The cone operator b defined in Definition 3.1.10 is a chain homotopy on $LC_\bullet(Y)$ connecting the identity map to the zero map, i.e. $\partial b + b\partial \ = \mathbb{1}$.

Proof. Since we are considering convex subsets of Euclidean space. One has:

$$\begin{aligned}\partial b([w_0, \cdots, w_n]) &= \partial[b, w_0, \cdots, w_n] \\ &= [w_0, \cdots, w_n] - \sum_{i=0}^{n}(-1)^i[b, w_0, \cdots, \hat{w}_i, \cdots, w_n] \\ &= [w_0, \cdots, w_n] - \sum_{i=0}^{n}(-1)^i b[w_0, \cdots, \hat{w}_i, \cdots, w_n] \\ &= [w_0, \cdots, w_n] - b\Bigg(\sum_{i=0}^{n}(-1)^i \ [w_0, \cdots, \hat{w}_i, \cdots, w_n]\Bigg) \\ &= [w_0, \cdots, w_n] - b\Bigg(\partial[w_0, \cdots, w_n]\Bigg).\end{aligned}$$

So

$$\partial b([w_0, \cdots, w_n]) = [w_0, \cdots, w_n] - b(\partial[w_0, \cdots, w_n]).$$

Hence $\partial b + b\partial \ = \mathbb{1}$. □

3.1.13 Remark. To avoid having the particular case for example 0-simplices, it will be good to set $LC_{-1}(Y) = \mathbb{Z}$ generated by the empty simplex $[\emptyset]$, with $\partial[w_0] = [\emptyset]$ for all 0-simplices $[w_0]$.

3.1.14 Definition. The subdivision operator on linear chains $S : LC_n(Y) \longrightarrow LC_n(Y)$ is defined inductively as follows:
Let $\lambda : \varDelta^n \longrightarrow Y$ be a generator of $LC_n(Y)$ and let b_λ be the image of the barycenter of $\varDelta^n$ under λ, i.e. $b_\lambda = \lambda(barycenter \quad of \quad \varDelta^n)$. Then we set $S([\emptyset]) = [\emptyset]$, and $S(\lambda) = b_\lambda(S\partial\lambda)$ where $b_\lambda : LC_{n-1}(Y) \longrightarrow LC_n(Y)$ is the cone operator defined earlier. More precisely, the induction is given by $S_n(\lambda) = b_\lambda(S_{n-1}\partial\lambda)$.

3.1.15 Lemma. The subdivision operator $S : C_n(X) \longrightarrow C_n(X)$ is chain map.

Proof. Let us check that the maps S satisfy $\partial S = S\partial$. We want to check the cases $n = 0$ and $n = -1$. In fact, S is the identity on $LC_0(Y)$ and $LC_{-1}(Y)$, in fact by $S\lambda = b_\lambda S\lambda = b_\lambda S[\emptyset] = b_\lambda[\emptyset]$. Then we use $\partial b_\lambda \lambda + b_\lambda \partial \lambda = \lambda$ and induction

$$\begin{aligned} \partial S\lambda &= \partial(b_\lambda(S\partial\lambda)) \\ &= S\partial\lambda - b_\lambda(\partial S\partial\lambda) \\ &= S\partial\lambda - b_\lambda S(\partial\partial\lambda) \\ &= S\partial\lambda. \end{aligned}$$

That implies $\partial S = S\partial$ on $LC_0(Y)$.

The result for larger n is given by the following calculation.

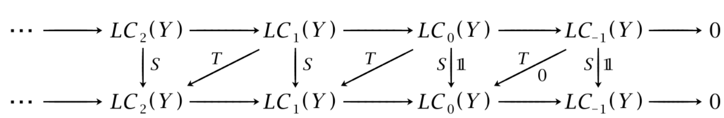
$$\begin{aligned} \partial S\lambda &= \partial b_\lambda(S\partial\lambda) \\ &= S\partial\lambda - b_\lambda(\partial S\partial\lambda) \quad \text{since} \quad \partial b_\lambda = \mathbb{1} - b_\lambda\partial \\ &= S\partial\lambda - b_\lambda S(\partial\partial\lambda) \qquad \text{since} \qquad \partial S(\partial\lambda) = S\partial(\partial\lambda) \quad \text{by induction on} \quad n \\ &= S\partial\lambda \quad \text{since} \quad \partial\partial = 0. \end{aligned}$$

□

3.1.16 Example. Let $\lambda = [v_0, v_1] : \Delta^1 \to Y$. Then;

$$\begin{aligned} S(\lambda) = b_\lambda S(\partial\lambda) &= b_\lambda S([v_1] - [v_0]) \\ &= b_\lambda([v_1] - [v_0]) \\ &= [b_\lambda, v_1] - [b_\lambda, v_0]. \end{aligned}$$

3.1.17 Definition. We define a chain homotopy $T : LC_n(Y) \longrightarrow LC_{n+1}(Y)$ between S and the identity, fitting into a diagram



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such that T on $LC_n(Y)$ is given inductively by setting $T = 0$ for $n = -1$ and letting $T\lambda = b_\lambda(\lambda - T\partial\lambda)$ for $n \geq 0$. More precisely, the induction is given by $T_n\lambda = b_\lambda(\lambda - T_{n-1}\partial_n\lambda)$.

3.1.18 Lemma. The map T is a chain homotopy between S and the identity.

Proof. We verify the formula on $LC_n(Y)$ with $n \geq 0$ by doing the following calculation.

$$\begin{aligned} \partial T\lambda &= \partial b_\lambda(\lambda - T\partial\lambda) \\ &= \lambda - T\partial\lambda - b_\lambda\partial(\lambda - T\partial\lambda) \quad \text{since} \quad \partial b_\lambda = \mathbb{1} - b_\lambda\partial \\ &= \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \partial T(\partial\lambda)) \quad \text{by induction on} \quad n \\ &= \lambda - T\partial\lambda - b_\lambda\big[S(\partial\lambda) + T\partial(\partial\lambda)\big] \\ &= \lambda - T\partial\lambda - S\lambda \quad \text{since} \quad \partial\partial = 0 \quad \text{and} \quad S\lambda = b_\lambda(S\partial\lambda). \end{aligned}$$

Now we can discard the group $LC_{-1}(Y)$ and the relation $\partial T + T\partial = \mathbb{1} - S$ still holds since T was zero on $LC_{-1}(Y)$. □

3.1.19 Lemma. Let us consider

$$\begin{aligned} S:\ C_n(X) &\to C_n(X) \\ \sigma &\mapsto \sigma_\sharp S\Delta^n \end{aligned}$$

for a singular n simplex $\sigma : \Delta^n \longrightarrow X$, where $S\Delta^n$ is the sum of the n simplices in the barycentric subdivision of Δ^n, with certain signs, $S\sigma$ is the corresponding signed sum of the restrictions of σ to the n simplices of the barycentric subdivision of Δ^n, and $\sigma_\sharp$ is the induced homomorphism. Then the operator S is a chain map.

Proof. For the proof we use Lemma 3.1.15, so

$$\begin{aligned} \partial S\sigma &= \partial\sigma_\sharp S\Delta^n = \sigma_\sharp\partial S\Delta^n = \sigma_\sharp S\partial\Delta^n \\ &= \sigma_\sharp S(\sum_i(-1)^i\Delta^n_i) \quad \text{where} \quad \Delta^n_i \quad \text{is the} \quad i^{th} \quad \text{face of} \quad \Delta^n \\ &= \sum_i(-1)^i\sigma_\sharp S\Delta^n_i \\ &= \sum_i(-1)^i S(\sigma|\Delta^n_i) \\ &= S(\sum_i(-1)^i\sigma|\Delta^n_i) \\ &= S(\partial\sigma). \end{aligned}$$

□

3.1.20 Lemma. The map $T : C_n(X) \longrightarrow C_{n+1}(X)$ defined by $T\sigma = \sigma_\sharp T\Delta^n$, and this gives a chain homotopy between S and the identity.

Proof. Since the formula $\partial T + T\partial = \mathbb{1} - S$ holds by the calculation

$$\begin{aligned} \partial T\sigma = \partial\sigma_\sharp T\Delta^n = \sigma_\sharp\partial T\Delta^n = \sigma_\sharp(\Delta^n - S\Delta^n - T\partial\Delta^n) &= \sigma - S\sigma - \sigma_\sharp T\partial\Delta^n \\ &= \sigma - S\sigma - T(\partial\sigma), \end{aligned}$$

where the last equality follows just as in the previous displayed calculation, with S replaced by T. □

3.1.21 Lemma. Let S be as in Lemma 3.1.19, and let $m \in \mathbb{N}$. A chain homotopy between $\mathbb{1}$ and the iterate S^m is given by the operator

$$D_m = \sum_{i=0}^{m} TS^i.$$

Proof. We have

$$\begin{aligned}\partial D_m + D_m\partial &= \sum_{i=o}^{m}(\partial TS^i + TS^i\partial) = \sum_{i=o}^{m}(\partial TS^i + T\partial S^i)\\ &= \sum_{i=o}^{m}(\partial T + T\partial)S^i = \sum_{i=o}^{m}(\mathbb{1} - S)S^i\\ &= \sum_{i=o}^{m}(S^i - S^{i+1})\\ &= \mathbb{1} - S^m.\end{aligned}$$

□

3.2 The Key Isomorphism

3.2.1 Definition (Interior)**.** Let X be a topological space and $A \subset X$ a subspace. The interior of A denoted by int(A) is the union of all open subsets of X which are contained in A.

3.2.2 Definition. Let X be a topological space, $\mathcal{U} = \{U_i : i \in I\}$ a collection of subspaces of X such that $\{\text{int}(U_i) : i \in I\}$ forms an open cover of X, i.e.

$$\bigcup_{i\in I}\text{int}(U_i) = X.$$

3.2.3 Definition ($C_n^{\mathcal{U}}(X)$ and $H_n^{\mathcal{U}}(X)$)**.** [16] The complex $C_\bullet^{\mathcal{U}}(X) \subseteq C_\bullet(X)$ is the subgroup generated by those singular n-simplices $\sigma : \Delta^n \longrightarrow X$ such that $\sigma(\Delta^n) \subseteq U_i$ for some $i \in I$, i.e.

$$C_n^{\mathcal{U}}(X) = \Big\{ \textstyle\sum_i n_i\sigma_i \in C_n(X) : n_i \in \mathbb{Z} \quad \sigma_i : \Delta^n \to X; \text{such that there exists } j \text{ with} \quad \sigma_i(\Delta^n) \subset U_j \Big\}.$$

And we define : $H_n^{\mathcal{U}}(X)$ as $H_n(C_\bullet^{\mathcal{U}}(X))$.

3.2.4 Remark. So given $C_n(X) \xrightarrow{\partial} C_n(X)$ and $C_n^{\mathcal{U}}(X) \xrightarrow{\partial} C_n^{\mathcal{U}}(X)$ we have a chain complex $(C_n^{\mathcal{U}}(X), \partial)$ which is in fact a subcomplex of $(C_n(X), \partial)$.

The following proposition says that the natural map $H_n^{\mathcal{U}}(X) \to H_n(X)$ is an isomorphism for all n.

3.2.5 Proposition. The inclusion $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence, that is, there is a chain map $\rho : C_n(X) \longrightarrow C_n^{\mathcal{U}}(X)$ such that $\iota \circ \rho$ and $\rho \circ \iota$ are chain homotopic to the identity. Hence ι induces isomorphisms $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n.

Proof. The proof is based on the preceding lemmas.

The first thing to do is to construct $\rho : C_n(X) \longrightarrow C_n^{\mathcal{U}}(X)$. We will proceed in two steps :

- First step: Let $\sigma : \mathit{\Delta}^n \longrightarrow X$ be a singular n-simplex in X. If $U_i, i \in I$ is an open sets in $\mathcal{U}$, then $\sigma^{-1}(\text{int}(U_i))$, is an open cover of $\mathit{\Delta}^n$ which has a Lebesgue number δ. Then there is $m \in \mathbb{N}$ such that all non-trivial summands of $S^m(\Delta)^n$ have diameter $< \delta$. This means that they are mapped to one of the sets U_i. Let $m(\sigma)$ be the smallest m such that each simplex appearing in $S^m(\Delta^n)$ is mapped to one of the sets U_i by σ (i depends on the simplex). Then $\partial\rho(\sigma)$ is in $C^{\mathcal{U}}_\bullet(X)$ but the subdivision might not be optimal. Thus it is not clear that ρ is a chain map.

- Second step: Let σ and $m(\sigma)$ be as in the first step and we define:

$$\begin{aligned} D:\ C_n(X) &\to C_n(X) \\ \sigma &\mapsto D_{m(\sigma)}\sigma \end{aligned}$$

We have already

$$\underbrace{\partial D_{m(\sigma)}\sigma}_{=D\sigma} + \underbrace{D_{m(\sigma)}(\partial\sigma)}_{\neq D_{m(\sigma)}} = \sigma - S^{m(\sigma)}\sigma.$$

Reorganizing the summands

$$\partial D\sigma + D\partial\sigma = \sigma - \underbrace{(S^{m(\sigma)} + D_{m(\sigma)}\partial\sigma - D(\partial\sigma)}_{=:\rho}.$$

Therefore, we define

$$\begin{aligned} \rho : C_\bullet(X) &\longrightarrow C_\bullet(X) \\ \sigma &\mapsto S^{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma - D\partial\sigma. \end{aligned}$$

It can be interpreted in terms of that equation:

$$\rho = \mathbb{1} - \partial D - D\partial.$$

It follows easily that ρ is a chain map since:

$$\partial\rho(\sigma) = \partial\sigma - \partial^2 D\sigma - \partial D\partial\sigma = \partial\sigma - \partial D\partial\sigma$$

and

$$\rho(\partial\sigma) = \partial\sigma - \partial D\partial\sigma - D\partial^2\sigma = \partial\sigma - \partial D\partial\sigma.$$

To prove that $\rho(\sigma) \in C^{\mathcal{U}}_n(X)$, we compute $\rho(\sigma)$ more explicitly:

$$\begin{aligned} \rho(\sigma) &= \sigma - \partial D_\sigma - D(\partial\sigma) \\ &= \sigma - \partial D_{m(\sigma)}\sigma - D(\partial\sigma) \\ &= S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \quad \text{since} \quad \partial D_m + D_m\partial = \mathbb{1} - S^m. \end{aligned}$$

We have $S^{m(\sigma)}\sigma \in C^{\mathcal{U}}_n(X)$ by the definition of $m(\sigma)$. The terms $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ are linear combinations of terms $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ for σ_j the restriction of σ to the j^{th} face of Δ^n, then $m(\sigma_j) \leq m(\sigma)$, so every term $TS^i(\sigma_j)$ in $D(\partial\sigma)$ will be a term in $D_{m(\sigma)}(\partial\sigma)$. Hence the difference $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ consists of terms $TS^i(\sigma_j)$ with $\iota \geq m(\sigma_j)$, and these terms are in $C^{\mathcal{U}}_n(X)$ by definition of T. In fact, T takes element from $C^{\mathcal{U}}_{n-1}(X)$ to $C^{\mathcal{U}}_n(X)$.

We have shown that, $\iota \circ \rho : C_\bullet(X) \longrightarrow C_\bullet(X)$ is chain homotopic to $\mathbb{1}$. By the definition of ρ it follows that if $\sigma \in C^{\mathcal{U}}_n(X)$, then $m(\sigma) = 0$. Hence, $S^{m(\sigma)}\sigma = \sigma$ and $\rho \circ \iota : C_\bullet(X) \longrightarrow C_\bullet(X)$ is chain homotopic to $\mathbb{1}$ (because $\rho \circ \iota = \mathbb{1}$). Thus we have shown that ρ is a chain homotopy inverse for ι. Hence ι induces isomorphisms $H^{\mathcal{U}}_n(X) \cong H_n(X)$ for all n. □

3.3 Establishing the Mayer-Vietoris Theorem

We now move on to stating and proving the Excision theorem. And then we will prove by the Mayer-Vietoris sequences.

3.3.1 Theorem (Excision theorem). *[10] For subspaces $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$ for all n.*

Proof. We use Proposition 3.2.5, so we for a space X, let us consider $\mathcal{U} = \{A, B\}$ be a collection of subspaces of X whose interiors form an open cover of X, then $X = A \cup B$. We also set $C_n^{\mathcal{U}}(X) := C_n(A+B)$ to be the sums of chains in A and chains in B. So we have the formula $\partial D + D\partial = \mathbb{1} - \iota \circ \rho$ and $\rho \circ \iota = \mathbb{1}$. All the maps that are in these formulas take chains in A to chains in A, so they induce quotient maps. Since the quotient map satisfied the formulas $\partial D + D\partial = \mathbb{1} - \iota \circ \rho$ and $\rho \circ \iota = \mathbb{1}$. So the inclusion $C_n(A + B)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$ induces an isomorphism on homology. Now using the inclusion $(B, A \cap B) \hookrightarrow (X, A)$, we get the map $C_n(B)/C_n(A \cap B) \longrightarrow C_n(A + B)/C_n(A)$ which induce an isomorphisms on homology. Since the chain complexes are isomorphic: chains that are in B which do not lie in $A \cap B$ are the same as chains that are either entirely in A or entirely in B but are not in A. So we get the isomorphism $H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$ induced by inclusion.

□

Now we present the Mayer-Vietoris sequence which is a powerful tool to help compute the homology of topological spaces.

3.3.2 Theorem (Mayer-Vietoris sequences). *[10]*

Let X be a topological space, and let $A, B \subset X$ be subspaces such that X is the union of the interiors of A and B. Then there exists a long exact sequence

$$\cdots \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0 \tag{3.3.1}$$

Proof. Let X be a topological space, and let $A, B \subset X$ be subspaces such that X is the union of the interiors of A and B and let $\mathcal{U} = \{A, B\}$. Let $C_n(A+B)$ denote the subgroup of $C_n(X)$ whose elements are precisely sums of singular simplices in either A or B. The boundary maps ∂ on $C_n(X)$ restrict to $C_n^{\mathcal{U}}(X) := C_n(A + B)$ and we get a chain complex $(C_\bullet(A + B), \partial)$.

According to Proposition 3.2.5, the inclusions $C_n^{\mathcal{U}}(X) := C_n(A + B) \hookrightarrow C_n(X)$ induce isomorphisms on homology groups. So we have

$$H_n^{\mathcal{U}}(X) \cong H_n(X). \tag{3.3.2}$$

To this end, for $n \in \mathbb{N}, \geq 0$, consider the following sequence:

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \to 0 \tag{3.3.3}$$

where, $\phi(x) = (x, -x)$ for all $x \in C_n(A \cap B)$ and $\psi(x, y) = x + y$ for all $(x, y) \in C_n(A) \oplus C_n(B)$. We claim that this sequence is exact, in fact:

- ψ is surjective by the definition of $C_n(A + B)$.

- ϕ is injective, since Ker$f\phi = 0$.
- For all $x \in C_n(A \cap B), \psi \circ \phi(x) = x - x = 0$. Therefore Im$(\phi) \subseteq$ Ker$f(\psi)$.
- If $(x, y) \in$ Ker(ψ), then x is a chain in A, y is a chain in B, and $y = -x$. This implies that x is a chain in $A \cap B$ and $\phi(x) = (x, -x) = (x, y)$. Therefore Ker$f(\psi) \subseteq$ Im(ϕ).

Now applying the theorem of long exact sequence in homology 2.3.8 to the sequence given in the equation 3.3.3, we get

$$\cdots \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n^{\mathcal{U}}(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0, \quad (3.3.4)$$

Plugging these isomorphisms $H_n^{\mathcal{U}}(X) \cong H_n(X)$ (equation 3.3.2) into the above long exact sequence, we get the Mayer-Vietoris Sequence which is then the long exact sequence associated to the short:

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \to 0.$$

□

3.3.3 Remark (The boundary operator ∂ is well defined)**.** In fact, every element $\alpha \in H_n(X)$ is represented by a cycle z and by the barycentric subdivision 3.1.5, we can pick $z = x + y$ in such a way that x supported in A and y supported in B. We have $\partial x = -\partial y$ since $\partial(x + y) = 0$ by definition of the sequence 3.3.3, and the element $\partial\alpha \in H_{n-1}(A \cap B)$ is represented by the cycle $\partial x = -\partial y$.

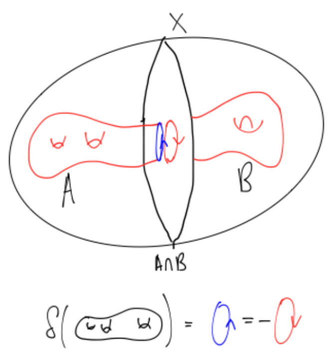

Figure 3.4: Representation of the boundary map ∂.

3.3.4 Remark (Mayer-Vietoris sequences follow also from excision theorem 3.3.1)**.** Another way of proving Mayer-Vietoris sequences 3.3.2 is to use directly the Excision theorem 3.3.1. In fact we consider the maps φ and ψ to be given in terms of inclusion. On the other hand, the map ∂ comes from the composition of the maps ∂_1 and ∂_2 satisfying :

$$H_n(X) \xrightarrow{\partial_1} H_n(X, A) \cong H_n(B, B \cap A) \xrightarrow{\partial_2} H_{n-1}(A \cap B)$$

where ∂_2 comes from the exactness sequence. And the isomorphism $H_n(X, A) \cong H_n(B, B \cap A)$ comes from Excision theorem, so it just remains to check that the sequence is exact. So it turns out that the **Mayer-Vietoris sequence theorem** is another expression of the **Excision theorem**.

3.3.5 Remark (Reduced version of the Mayer-Vietoris sequence)**.** For reduced homology 2.4.2 there is also a Mayer-Vietoris sequence for reduced homology, under the assumption that A and B have non-empty intersection. The sequence is identical for positive dimensions and ends as:

$$\cdots \to \tilde{H}_0(A \cap B) \to \tilde{H}_0(A) \oplus \tilde{H}_0(B) \to \tilde{H}_0(X) \to 0.$$

Having studied Mayer-Vietoris sequences, we will now use it to compute and describe the homology group of some spaces in the next chapter.

4. Applications

In this chapter, we use Mayer-Vietoris sequences to do lots of computations. In the first section, we compute the homology of spheres which leads us to many results, including the Brouwer fixed point theorem. And then we compute the homology of the wedge of two spaces in section 4.2. Section 4.3 deals with the torus. Furthermore in section 4.4, we calculate the homology of the Klein bottle. Finally, Section 4.5 deals with the homology of the real projective plan.

4.1 Homology of the Spheres

4.1.1 Definition. For $n \in \mathbb{N}, \geq 0$, n-sphere (unit sphere) in $\mathbb{R}^{n+1}$ is defined by:

$S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$.

The case when $n = 0$ is trivial. In fact since S^n has two path components, so $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$. Further, we have $H_p(S^0) \cong 0, p > 0$, because each path component is a single point.

The next thing we are going to do is to focus on S^1.

4.1.2 Theorem. *We have*

$$H_p(S^1) = \begin{cases} \mathbb{Z} & \textit{if} \quad p \in \{0, 1\} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof.

- The case $p = 0$, is straightforward $H_0(S^1) = \mathbb{Z}$, since S^1 is path connected.
- Let now consider the case $p = 1$, let $A = S^1 - \{(0, 1)\}$ and $B = S^1 - \{(0, -1)\}$ be to two open subsets of S^1, we have both A and B are open in S^1 and hence $A =$ intA and $B =$ intB. Further, intA $\cup$ intB $= S^1$. We claim that A and B are homeomorphic to $\mathbb{R}$, and the homeomorphism is given by the stereographic projection in [7]. So $\mathbb{R}$ being contractile, it has the same type of homotopy as a point i.e.

$$H_p(A) = H_p(B) = H_p(\mathbb{R}) = \begin{cases} \mathbb{Z} & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p \neq 0. \end{cases}$$

Also, $A \cap B$ is homotopy equivalent to the 0-sphere S^0 so $H_p(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$ because we have two connected components for $p = 0$ and we have $H_p(A \cap B) = 0$ for $p \neq 0$. Now using the reduced Mayer-Vietoris sequence, we have :

$$\cdots \to \tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(A \cup B) \to \tilde{H}_0(A \cap B) \to \cdots$$

which implies:

$$0 \to \tilde{H}_1(S^1) \to \mathbb{Z} \to 0.$$

So $\tilde{H}_1(S^1) \cong \mathbb{Z}$ which implies $H_1(S^1) \cong \mathbb{Z}$.

- Let us consider $p \geq 2$, applying the Mayer-Vietoris sequences we will have:

$$0 \to H_p(S^1) \to H_{p-1}(A \cap B) \to 0.$$

Since $H_{p-1}(A \cap B) = 0$, it follows by the exactness that $H_p(S^1) = 0$. This ends the proof. So for all $p \geq 2$, $H_p(S^1) = 0$.

□

Now we are ready to handle the general case.

4.1.3 Theorem. *For any $n \geq 1$, we have*

$$H_p(S^n) = \begin{cases} \mathbb{Z} & \text{if} \quad p = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We prove the result by induction on n. The case $n = 0$ or $n = 1$ we are done by the preceding theorems. Let $n > 1$, we again cut up S^n as $A = S^n - \{N\}$ and $B = S^n - \{S\}$ where N and S are the north and south poles of S^n. We examine the Mayer-Vietoris sequence with $p > 0$

$$\cdots \to H_p(A) \oplus H_p(B) \to H_p(S^n) \to H_{p-1}(A \cap B) \to H_{p-1}(A) \oplus H_{p-1}(B) \to \cdots.$$

If $p > 1$ the end terms are isomorphic to 0, because A and B are homeomorphic to $\mathbb{R}^n$, and the homeomorphism is given by the stereographic projection in [7]. Further $H_{p-1}(A \cap B) \cong H_{p-1}(S^{n-1})$ because $A \cap B \cong \mathbb{R} \times S^{n-1} \simeq S^{n-1}$ is homotopy equivalent to $\mathbb{R}^n - \{0\}$. Using the inductive hypothesis and the exactness we get $H_p(S^n) \cong \mathbb{Z}$ if $p = n$ and $H_p(S^n) = 0$ otherwise.

If $p = 1$, we use the reduced version of the Mayer-Vietoris sequence, then we will get:

$$0 \to \tilde{H}_1(S^n) \to \tilde{H}_0(S^{n-1}) \to 0.$$

So $\tilde{H}_1(S^n) \cong \tilde{H}_0(S^{n-1}) \cong 0$. That implies $H_1(S^n) \cong 0$.

□

4.1.4 Corollary. For $m \neq n$ the spheres S^m and S^n are not homotopy equivalent.

Proof. Let us assume $n \neq m$, so by the preceding Theorem 4.1.3, the homology groups $H_n(S^m)$ and $H_n(S^n)$ are not the same, it follows that the homology groups of S^m and S^n are not isomorphic. So, S^m and S^n are not homotopy equivalent [7]. □

4.1.5 Corollary. For $n \geq 0$ the sphere S^n is not contractible.

Proof. Let us assume $n \geq 0$, by contradiction, assume that S^n is contractible i.e. $S^n \simeq \{*\}$. So

$$H_p(S^n) = H_p(\{*\}) = \begin{cases} \mathbb{Z} & \text{if} \quad p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let us take $n = p$ so we have two cases:

If $n = 0$, In that case, we have $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$, which contradicts the fact that $S^n \simeq \{*\}$.

Now if $n \neq 0$ we have $H_p(S^n) = 0$, that is a contradiction, because using Theorem 4.1.3, we have $H_p(S^n) = \mathbb{Z}$ if $p = n \neq 0$. Hence S^n is not contractible. □

4.1.6 Corollary. (Invariance of dimension). Let $m, n \geq 0$. The spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic if and only if $m = n$.

Proof. Let us assume that $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, so we have a homeomorphism

$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$. Let us choose an arbitrary $x_0 \in \mathbb{R}^m$. We obtain an induced homeomorphism

$f : \mathbb{R}^m - \{x_0\} \to \mathbb{R}^n - \{f(x_0)\}$. If one of n and m are zero, automatically the other have to be zero due to the homeomorphism. Let us assume that $n, m > 0$. Then $\mathbb{R}^m - \{x_0\}$ and the $\mathbb{R}^n - \{f(x_0)\}$ are homotopy equivalent to a sphere of the respective dimension i.e. $\mathbb{R}^m - \{x_0\} \backsimeq S^{m-1} \backsimeq \mathbb{R}^n - \{f(x_0)\} \backsimeq S^{n-1}$. Which implies the homotopy equivalence $S^{-1} \backsimeq S^{n-1}$. But by Corollary 4.1.4, we have $n = m$. That leads us to a contradiction. If we suppose now $n = m$, it is straightforward that $\mathbb{R}^m$ is homeomorphic to itself. □

4.1.7 Definition. For all $n \in \mathbb{N}, \geq 0$, n-ball (unit ball) in $\mathbb{R}^n$ is defined to be :

$$D^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \leq 1\}.$$

4.1.8 Corollary. (Brouwer fixed point theorem). For $n \geq 0$, every continuous map $f : D^n \longrightarrow D^n$ has a fixed point.

Proof. If $n = 0$, we are done obvious since D^0 is the one-point space. By contradiction let us assume that f has no fixed point. Let us consider a retraction $r : D^n \longrightarrow S^{n-1} = \partial D^n$ defined by taking the intersection of the ray from $f(x)$ through x with ∂D^n which is continuous.

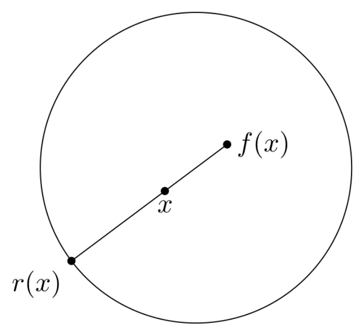

Figure 4.1: The disk D^2.

If $x \in \partial D^n$ then $r(x) = x$. So we have

$S^{n-1} = \partial D^n \xrightarrow{i} D^n \xrightarrow{r} \partial D^n = S^{n-1}$ where $r \circ i = \mathbb{1}_{S^{n-1}}$. That induces on homology

$H_{n-1}(S^{n-1}) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(S^{n-1})$ where $r \circ i = \mathbb{1}_{S^{n-1}}$.

Since D^n is contractile we have $H_{n-1}(D^n) = 0$ and using Theorem 4.1.3, we also have $H_{n-1}(S^{n-1}) = \mathbb{Z}$. So we get $\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{r_*} \mathbb{Z}$ which implies the contraction, since r is a retraction. So f has a fixed point. □

So far, we have been talking about spheres. We now move on to something different.

4.2 Homology of the Wedge of Spaces

4.2.1 Definition (Wedge Sum). [10] Let X and Y be two topological spaces. Let $x \in X$ and $y \in Y$ be basepoints respectively of X and Y. The wedge sum of X and Y (with respect to x and y) is the quotient $X \vee Y = X \sqcup Y/\sim$ of the disjoint union $X \sqcup Y$ by the smallest relation $\sim$ identifying x and y to a single basepoint.

4.2.2 Theorem. *Let X and Y be two topological spaces in which the basepoints of X and Y are identified in $X \vee Y$ and are deformation retracts of neighborhoods of X and of Y.*

Then $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for all $n \in \mathbb{N}^*$.

Proof. Let x_0 be the basepoint of $X \vee Y$, with the neighborhoods of $A \subseteq X$ and $B \subseteq Y$ so that $X \vee Y = (X \cup B) \cup (Y \cup A)$. The Mayer-Vietoris sequence

$$\cdots \to H_n\big[(X \cup B) \cap (Y \cup A)\big] \to H_n(X \cup B) \oplus H_n(Y \cup A) \to H_n(X \vee Y) \to H_{n-1}\big[(X \cup B) \cap (Y \cup A)\big] \to \cdots$$

Using the fact that A and B are deformation retract onto x_0, we have $X \cup B \simeq X$ and $Y \cup A \simeq Y$. So $H_n(X \cup B) \cong H_n(X)$ and $H_n(Y \cup A) \cong H_n(Y)$, moreover $(X \cup B) \cap (Y \cup A) = A \cup B$. That leads us to the following exact sequence:

$$\cdots \to H_n(A \cup B) \to H_n(X) \oplus H_n(Y) \to H_n(X \vee Y) \to H_{n-1}(A \cup B) \to \cdots .$$

But we have taken a neighborhoods of x_0, such that it satisfied $A \cup B$ is contractible, so $H_n(A \cup B) = 0$ for all $n \in \mathbb{N}^*$. Then we end up getting :

$$0 \to H_n(X) \oplus H_n(Y) \to H_n(X \vee Y) \to 0.$$

Hence $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for $n \in \mathbb{N}^*$.

□

4.2.3 Remark. In particular, we have the same relation for the reduced homology, i.e.

$$\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$$

consequently, we can say that the homology functor $\tilde{H}_* : Top \to GrAb$ from the category of pointed topological spaces to the category of graded abelian groups is additive.

4.2.4 Corollary. Let S^1 be a unit circle. Let us consider $S^1 \vee S^1$ the Wedge sum of two circles S^1 and S^1 with the common basepoint $x_0 \in S^1$.

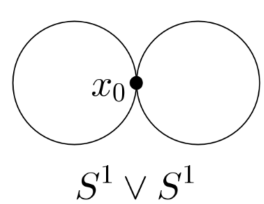

Figure 4.2: The wedge of two copies of the circle.

Then:

$$H_n(S^1 \vee S^1) = \begin{cases} \mathbb{Z} & \text{if} \quad n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \geq 2. \end{cases}$$

Proof. So the case $n = 0$ comes from the fact that the wedge $S^1 \vee S^1$ is connected. For the case $n = 1$, we consider the isomorphism $H_1(S^1 \vee S^1) \cong H_1(S^1) \oplus H_1(S^1)$ given in the preceding Theorem 4.2. So $H_1(S^1 \vee S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. For $n \geq 2$, we still $H_n(S^1 \vee S^1) \cong H_n(S^1) \oplus H_n(S^1)$ for $n \in \mathbb{N}, n \geq 2$. But we have from Theorem 4.1.2, $H_n(S^1) \cong 0$ for $n \geq 2$. Hence $H_n(S^1 \vee S^1) \cong 0$ for $n \in \mathbb{N}, n \geq 2$. □

4.3 Homology of the Torus

4.3.1 Definition (Torus). Let $X = [0,1] \times [0,1]$ the product of two copies of the interval, and let $\sim$ be the equivalence relation on X that is generated by the identifications $(s,0) \sim (s,1)$ for all $s \in [0,1]$ and $(0,t) \sim (1,t)$ for all $t \in [0,1]$. We write $T = X/\sim$ for the resulting quotient space and note that it is homeomorphic to a torus.

The representation is given by the figure below:

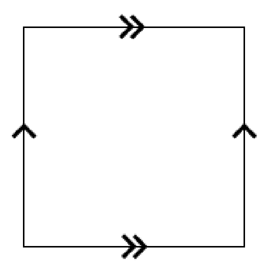

Figure 4.3: Torus representation.

4.3.2 Remark. The preceding representation comes from the fact that every compact surface admits a polygonal presentation (a planar diagram). In particular, for the Torus we have can see the figure below.

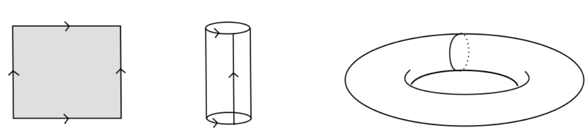

Figure 4.4: Reduction of the Torus [8].

4.3.3 Theorem. *The singular homology groups of T are as follows:*

$$H_n(T) = \begin{cases} \mathbb{Z} & \textit{if} \quad n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \textit{if} \quad n = 1 \\ \mathbb{Z} & \textit{if} \quad n = 2 \\ 0 & \textit{if} \quad n \geq 3. \end{cases}$$

Proof. Let $I = [0,1], A$ be an open square inside $I \times I$. Also let us consider B' be a closed square inside A, as in the figure below.

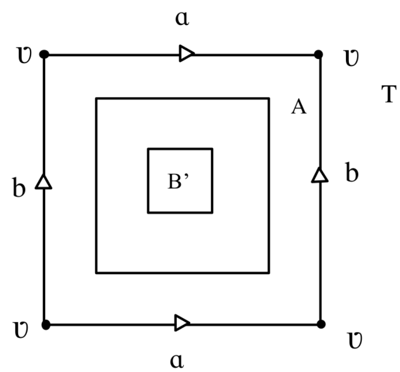

Figure 4.5: Torus.

We define $B = T - B'$. Since A is contractile, $A \backsimeq \{*\}$; $B \simeq \partial(I \times I)/\sim \simeq S^1 \vee S^1$ by identification, where $S^1 \vee S^1$ is the wedge of two circles. Moreover, we have $A \cap B \simeq S^1$. The Mayer-Vietoris sequence is:

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(T) \to H_{n-1}(A \cap B) \to \cdots \to H_0(T) \to 0.$$

Now, we proceed in three steps.

Case 1, $n \geq 3$, $H_n(A) = H_n(B) = H_n(A \cap B) = 0$, because A is contractible and B is homotopy equivalent to the wedge $S^1 \vee S^1$. So

$$0 \to 0 \to H_n(T) \to 0 \to 0 \to 0 \to \cdots.$$

So $H_n(T) = 0$.

Case 2, $n = 0$, since T is connected, we have $H_0(T) = \mathbb{Z}$.

Case 3, Let us now deal with the case $n = 1$ and $n = 2$, the reduced version of the Mayer-Vietoris sequence gives:

$0 \to \tilde{H}_2(T) \to \tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(T) \to 0$, we can rewrite it as

$$0 \xrightarrow{\varphi_1} \tilde{H}_2(T) \xrightarrow{\varphi_2} \mathbb{Z} \xrightarrow{\varphi_3} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varphi_4} \tilde{H}_1(T) \xrightarrow{\varphi_5} 0.$$

Where directly we have $\varphi_1 = \varphi_5 = 0$; but for $\varphi_3 : H_1(A \cap B) \to H_1(A) \oplus H_1(B)$, we use the fact that the generator of $H_1(A \cap B)$ is 1 which corresponds to a loops around B'. To defined φ_3, we need to use Figure 4.5. So we have $\varphi_3(1) = b + a - b - a = 0$. Finally φ_3 is defined by the generator as follows:

$$\begin{aligned} \varphi_3 : \mathbb{Z} &\longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ 1 &\longmapsto (0,0). \end{aligned}$$

Indeed, Im$\varphi_3 = 0$; and Ker$\varphi_3 = \mathbb{Z}$ by definition of φ_3. Now applying the first isomorphism theorem to φ_2. We get:
$\frac{H_2(T)}{\ker\varphi_2} \cong$ Im$\varphi_2 =$ Ker$\varphi_3 = \mathbb{Z}$, since Ker$\varphi_2 =$ Im$\varphi_1 = 0$. We have $H_2(T) = \mathbb{Z}$.
Let now look for the value of $H_1(T)$. We apply once more the first isomorphism theorem to φ_4 to get:

$$\frac{\mathbb{Z} \oplus \mathbb{Z}}{\text{Ker}\varphi_4} \cong \text{Im}\varphi_4, \tag{4.3.1}$$

because of the exactness, we have Ker$\varphi_4 =$ Im$\varphi_3 = 0$. Moreover Im$\varphi_4 =$ Ker$\varphi_5 = H_1(T)$. The equation 4.3.1 becomes $\frac{\mathbb{Z}\oplus\mathbb{Z}}{\{0\}} \cong H_1(T)$. Finally, $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Hence

$$H_n(T) = \begin{cases} \mathbb{Z} & \text{if} \quad n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if} \quad n = 1 \\ \mathbb{Z} & \text{if} \quad n = 2 \\ 0 & \text{if} \quad n \geq 3. \end{cases}$$

□

4.4 Homology of the Klein bottle

We want to compute the homology groups of the Klein bottle.

4.4.1 Definition (Klein bottle)**.** The Klein Bottle can be defined as the quotient space K of the square $[0,1] \times [0,1]$ modulo the equivalence relation $\sim$ that is generated by the identifications $(s,0) \sim (s,1)$ for all $s \in [0,1]$, and $(0,t) \sim (1, 1-t)$ for all $t \in [0,1]$.

The representation is given by the figure below:

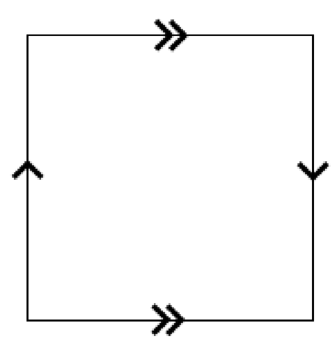

Figure 4.6: Klein bottle representation.

4.4.2 Remark. The representation comes from the fact that every compact surface admits a polygonal presentation (a planar diagram).

In particular, we can easily move from the polygonal representation to the surface as we can see for the case of the Klein bottle in the figure below.

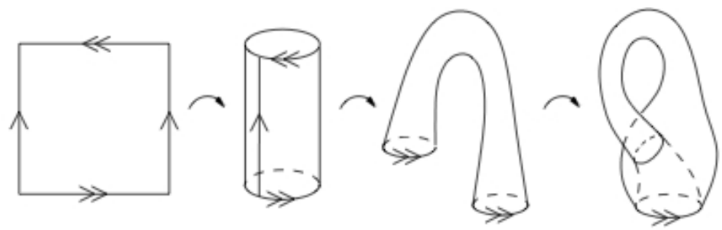

Figure 4.7: Reduction of the Klein bottle [8].

4.4.3 Theorem. *Let K be the Klein bottle. The singular homology groups of K are as follows:*

$$H_n(K) = \begin{cases} \mathbb{Z} & \text{if} \quad n = 0 \\ \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \geq 2. \end{cases}$$

Proof. Let $I = [0, 1], A$ be an open square inside $I \times I$. Also let us consider B' be a closed square inside A, as in the figure below:

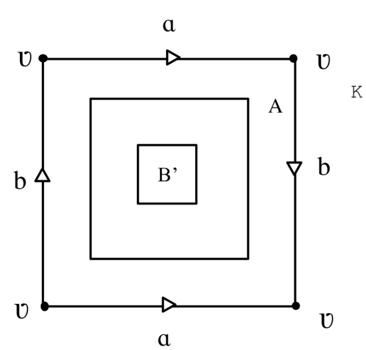

Figure 4.8: Klein bottle.

We define $B = K - B'$. Since A is contractile, $A \backsimeq \{*\}$; $B \simeq \partial(I \times I)/\sim \simeq S^1 \vee S^1$ by identification, where $S^1 \vee S^1$ is the wedge of two circles. Moreover, we have $A \cap B \simeq S^1$. By applying Mayer-Vietoris sequence we get :

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(K) \to H_{n-1}(A \cap B) \to \cdots \to H_0(K) \to 0.$$

Now, we proceed in three cases.

Case 1. $n \geq 3$, $H_n(A) = H_n(B) = H_n(A \cap B) = 0$, because A is contractible and B is homotopy equivalent to the wedge $S^1 \vee S^1$. So

$$0 \to 0 \to H_n(K) \to 0 \to 0 \to 0 \to \cdots .$$

So $H_n(K) = 0$.

Case 2. $n = 0$, since K is connected, we have $H_0(K) = \mathbb{Z}$.

Case 3. Let us now deal with the case $n = 1$, and $n = 2$, the reduced version of the Mayer-Vietoris sequence gives: $0 \to \tilde{H}_2(K) \to \tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(K) \to 0$, we can rewrite is as

$$0 \xrightarrow{\varphi_1} \tilde{H}_2(K) \xrightarrow{\varphi_2} \mathbb{Z} \xrightarrow{\varphi_3} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varphi_4} \tilde{H}_1(K) \xrightarrow{\varphi_5} 0.$$

Where directly we have $\varphi_1 = \varphi_5 = 0$; but for $\varphi_3 : H_1(A \cap B) \to H_1(A) \oplus H_1(B)$, we use the fact that the generator of $H_1(A \cap B)$ is 1 which corresponds to a loops around B'. To defined φ_3, we need to use Figure 4.8: So we have $\varphi_3(1) = b + a + b - a = 2b$. Finally φ_3 is defined by the generator as follows:

$$\begin{aligned} \varphi_3 : \mathbb{Z} &\longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ 1 &\longmapsto (0, 2). \end{aligned}$$

Indeed, Im$\varphi_3 = 0 \oplus 2\mathbb{Z}$; and Ker$\varphi_3 = 0$. Because φ_3 is injective. Now applying the first isomorphism theorem to φ_2. We get:
$\frac{H_2(K)}{\text{ker}\varphi_2} \cong$ Im$\varphi_2 =$ Ker$\varphi_3 = 0$, since Ker$\varphi_2 =$ Im$\varphi_1 = 0$. Hence $H_2(K) = 0$.
Let now look for the value of $H_1(K)$. We apply once more the first isomorphism theorem to φ_4 to get:

$$\frac{\mathbb{Z} \oplus \mathbb{Z}}{\text{Ker}\varphi_4} \cong \text{Im}\varphi_4, \tag{4.4.1}$$

because of the exactness, we have Ker$\varphi_4 =$ Im$\varphi_3 = 0 \oplus 2\mathbb{Z}$. Moreover Im$\varphi_4 =$ Ker$\varphi_5 = H_1(K)$. The equation 4.5.1 becomes $\frac{\mathbb{Z} \oplus \mathbb{Z}}{0 \oplus 2\mathbb{Z}} \cong H_1(K)$. Finally, $H_1(K) \cong \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$.

Hence

$$H_n(K) = \begin{cases} \mathbb{Z} & \text{if} \quad n = 0 \\ \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \geq 2. \end{cases}$$

□

4.5 Homology of the Projective Plane

In this section, we will use the Mayer-Vietoris sequence to compute the homology of real real projective plane $\mathbb{R}P^2$.

4.5.1 Definition. The real projective plan is made from the set of nonzero vectors $v \in \mathbb{R}^3 - \{0\}$ up to equivalence relation $v \sim \lambda v, \lambda \in \mathbb{R}^*$ under scaling i.e. $\mathbb{R}P^2 := \mathbb{R}^3 - \{0\}/\sim$. Moreover, $\mathbb{R}P^2$ it is equivalently identify as $S^2/\{-x \sim x\}$.

The representation is given by the figure below:

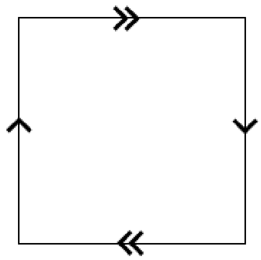

Figure 4.9: Projective plane $\mathbb{R}P^2$.

4.5.2 Theorem. *We have*

$$H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & \text{if} \quad n=0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if} \quad n=1 \\ 0 & \text{if} \quad n \geq 2. \end{cases}$$

Proof. The projective plane $\mathbb{R}P^2$ can be identified with the quotient $D^2/\sim$, where $\sim$ is the equivalence relation generated by $-x \sim x$ for every x in the boundary of the disk D^2. As we can see in the figure below:

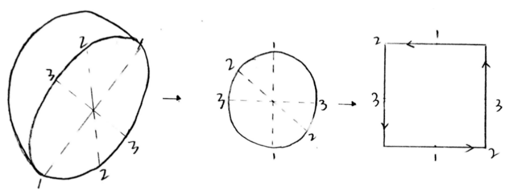

Figure 4.10: Reduction of $\mathbb{R}P^2$ [8].

So we set A to be a open disk inside D^2 and B' be a closed disk in A, as in the figure below:

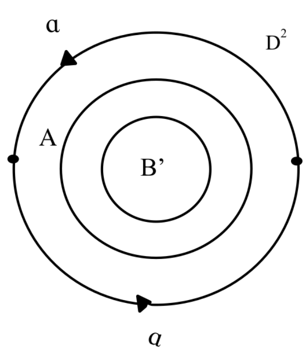

Figure 4.11: Projective plane.

We define $B = \mathbb{R}P^2 - B'$. Since A is contractile, $A \backsimeq \{*\}$; and we have $B \simeq S^1$. Moreover, we have $A \cap B \simeq S^1$. By applying Mayer-Vietoris sequence we get :

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(\mathbb{R}P^2) \to H_{n-1}(A \cap B) \to \cdots \to H_0(\mathbb{R}P^2) \to 0.$$

Now, we proceed in three cases.

Case 1, $n \geq 3$, $H_n(A) = H_n(B) = H_n(A \cap B) = 0$. So

$$0 \to 0 \to H_n(\mathbb{R}P^2) \to 0 \to 0 \to 0 \to \cdots.$$

So $H_n(\mathbb{R}P^2) = 0$.

Case 2, $n = 0$, since $\mathbb{R}P^2$ is connected, we have $H_0(\mathbb{R}P^2) = \mathbb{Z}$.

Case 3, Let us now deal with the case $n = 1$ and $n = 2$, the reduced version of the Mayer-Vietoris sequence gives: $0 \to \tilde{H}_2(\mathbb{R}P^2) \to \tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(\mathbb{R}P^2) \to 0$, we can rewrite is as

$$0 \xrightarrow{\varphi_1} \tilde{H}_2(\mathbb{R}P^2) \xrightarrow{\varphi_2} \mathbb{Z} \xrightarrow{\varphi_3} \mathbb{Z} \xrightarrow{\varphi_4} \tilde{H}_1(\mathbb{R}P^2) \xrightarrow{\varphi_5} 0.$$

Where directly we have $\varphi_1 = \varphi_5 = 0$; but for $\varphi_3 : H_1(A \cap B) \to H_1(A) \oplus H_1(B)$, we use the fact that the generator of $H_1(A \cap B)$ is 1 which corresponds to a loops around B'. To defined φ_3, we need to use Figure 4.11: So we have $\varphi_3(1) = a + a = 2a$. Finally φ_3 is defined by the generator as follows:

$$\begin{aligned} \varphi_3 : \mathbb{Z} &\longrightarrow \mathbb{Z} \\ 1 &\longmapsto 2. \end{aligned}$$

Indeed, Im$\varphi_3 = 2\mathbb{Z}$; and Ker$\varphi_3 = 0$. Because φ_3 is injective. Now applying the first isomorphism theorem to φ_2. We get:

$\frac{H_2(\mathbb{R}P^2)}{\ker\varphi_2} \cong$ Im$\varphi_2 =$ Ker$\varphi_3 = 0$, since Ker$\varphi_2 =$ Im$\varphi_1 = 0$. Hence $H_2(\mathbb{R}P^2) = 0$.

Let now look for the value of $H_1(\mathbb{R}P^2)$. We apply once more the first isomorphism theorem to φ_4 to get:

$$\frac{\mathbb{Z}}{\text{Ker}\varphi_4} \cong \text{Im}\varphi_4, \tag{4.5.1}$$

because of the exactness, we have Ker$\varphi_4 =$ Im$\varphi_3 = 2\mathbb{Z}$. Moreover Im$\varphi_4 =$ Ker$\varphi_5 = H_1(\mathbb{R}P^2)$. The equation 4.5.1 becomes $\frac{\mathbb{Z}}{2\mathbb{Z}} \cong H_1(\mathbb{R}P^2)$. Finally, $H_1(\mathbb{R}P^2) \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$.

Hence, we have the expected homology of the projective plane:

$$H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & \text{if} \quad n = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \geq 2. \end{cases}$$

□

5. Conclusion

The goal of this essay was to master the Mayer-Vietoris sequence and treat some applications. We first presented some notions in singular homology, and then we established one of the most important results in homological algebra [18], which says that, any short exact sequence of chain complexes in this form: $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ induces a long exact sequence in homology groups

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots .$$

Then we proved that for two elements cover $\{A, B\}$ of X there exists a short sequence

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \to 0.$$

So the Mayer-Vietoris sequence given below

$$\cdots \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0 \quad (5.0.1)$$

turned out to be the long exact sequence induced by it. After mastering very well the Mayer-Vietoris theorem (Theorem 3.3.2), we used it to make some calculations. In fact, we calculated the homology of many spaces, including the homology of the sphere. Actually, its homology yielded a lot of consequences that we had to clarify. One of them was the Brouwer fixed-point theorem, in fact, we justified why for $n \geq 0$, every continuous map $f : D^n \longrightarrow D^n$ has a fixed point. And also, we proved the following statement, "the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic if and only if $m = n$," known on the name, invariant of dimension theorem. We also discussed the problem of the non contractibility of the sphere. And we come to the conclusion with rigorous arguments that the sphere S^n is not contractible. From all these results, we can conclude that the tool Mayer-Vietoris sequence is very applicable. On the other hand, the homology of the wedge of spaces (Section 4.2) was also very rich in consequences, in the sense that for two spaces X and Y, we had the following isomorphism:

$$\tilde{H}_*(X \vee Y) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y),$$

which is very meaningful, actually it means that the homology functor $\tilde{H}_* : Top \to GrAb$ from the category of pointed topological spaces to the category of graded abelian groups is additive [11]. In the future, we plan to go further in that direction and try to provide the details of the fact that $\tilde{H}_*$ is excisive in the sense of Goodwillie calculus of functors [9]. So we will try to understand (by going over the definition of excisive in [19],) why the excisiveness of the functor $\tilde{H}_*$ is related to the fact that it has Mayer-Vietoris sequences for pushout squares [12].

Appendix A.

A.1 Some group notions

In this section, we recall some definitions and theorems on group theory for a well understanding of this essay.

A.1.1 Definition (Group). A group is a set G, together with an operation "$\cdot$" (called the group law of G) that combines any two elements a and b to form another element, denoted $a \cdot b$ or ab. Such that, the set and operation $(G, \cdot)$, satisfied four requirements known as the group axioms:

- For all a, b in G, the result of the operation $a.b$, is also in G. This axiom is called **closure.**
- For all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. This axiom is called **associativity.**
- There exists an element e in G such that, for every element a in G, the equation $e \cdot a = a \cdot e = a$ holds. This axiom is called **inverse element.**
- For each a in G, there exists an element b in G, commonly denoted a^{-1}, such that $a \cdot b = e$ and $b \cdot a = e$. This axiom is called **identity element.**

A.1.2 Remark. If moreover for all a, b in A; $a \cdot b = b \cdot a$, we say that $(G, \cdot)$ is an abelian group.

A.1.3 Theorem (First Isomorphic Theorem). *Let $\phi : G \to G'$ be a group homomorphism. Then*

$$G/\mathit{ker}(\phi) \cong \mathit{Im}(\phi).$$

Proof. We set $H = \ker(\phi)$, since $\ker(\phi)$ is a normal subgroup of G. This gives us a homomorphism $\psi : G/H \to \text{Im}(\phi)$ such that $\psi \circ \pi = \phi$, where $\pi : G \to G/H$ is the canonical surjection. Let $g' \in \text{Im}(\phi)$. Then there is a $g \in G$ such that $\phi(g) = g'$. But then $\psi([g]) = \phi(g) = g'$ and hence ψ is surjective. Suppose $\psi[g] = 0$. Then $\phi(g) = 0$, which means that $g \in \ker(\phi)$. But then $[g] = [0]$, which shows that ψ is injective. Thus ψ is an isomorphism. □

Here is a variation that is more used.

A.1.4 Theorem. *Let $\phi : G \to G'$ be a surjective group homomorphism. Then*

$$G/\mathit{ker}(\phi) \cong G'.$$

A.1.5 Definition (Free abelian group). [11] An abelian group G is free if there exists a subset $A \subseteq G$ (called a basis) such that every element $g \in G$ has a unique representation

$$g = \sum_{x \in A} n_x \cdot x$$

where n_x is an integer and equal to zero for all but finitely many $x \in A$.

A.1.6 Definition (Graded abelian group). A graded abelian group G is a collection of abelian groups $\{G_n\}_{n \geq 0}$. In particular, we call the graded group $\{H_n(X)\}_{n \geq 0}$ the homology of a topological space X.

Throughout this essay, we referred to the free abelian group with a given basis. The following result show that such a group always exists.

A.1.7 Theorem. *[11] Given an arbitrary set A there exists a free abelian group with basis A.*

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