GERSTENHABER AlGEBRAS AND THE HOMOLOGY OF SPACES OF LONG KNOTS AND LONG LINKS

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Gerstenhaber Algebras and the Homology of Spaces of Long Knots and Long Links

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Dédicace

Je dédie cet ouvrage

à mon père **Songhafouo**

et à ma mère **Dassi Colette**.

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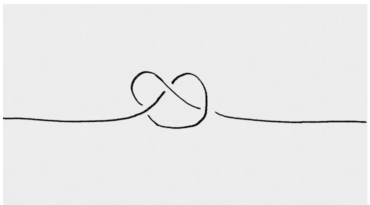
Contents

Introduction

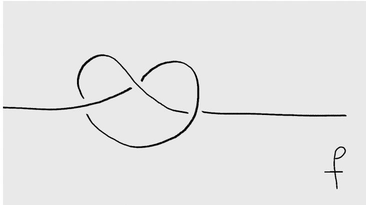
A *long knot* is a smooth embedding $f\colon \mathbb{R} \hookrightarrow \mathbb{R}^d$ from $\mathbb{R}$ to a Euclidean space $\mathbb{R}^d$, $d \geq 3$, that coincides outside a compact set with a fixed linear embedding (see Definition 2.1.1). The following is an example of a simple long knot.

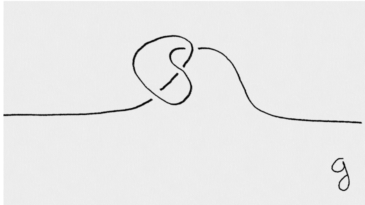


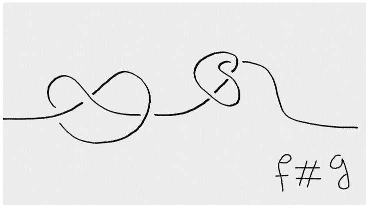
The *space of long knots*, denoted by $\mathcal{K}not$, is the collection of all long knots endowed with a suitable topology (see Definition 2.1.2). The goal of this thesis is to study the homology of that space. More precisely, we will explicitly compute its algebraic structure. Further in the work we will extend our study to spaces of *long links* and their *high dimensional analogues*, which are generalizations of long knots spaces.

The homology $H_*(\mathcal{K}not)$ has a very rich structure: we will show that it is a *Gerstenhaber algebra* (which can be viewed as a variation of a Poisson algebra). Roughly speaking, a *Gerstenhaber algebra* is a graded vector space $V = \bigoplus_k V_k$ endowed with a product $\times\colon V_p \otimes V_q \longrightarrow V_{p+q}$ and a bracket $\{-,-\}\colon V_p \otimes V_q \longrightarrow V_{p+q+1}$ satisfying some conditions (see Definition 3.2.3).

It is well known that there is a product on $\mathcal{K}not$ constructed as follows. Let us take two long knots, f and g, and "join" the end of one to the other. The result is a new long knot, $f\#g$, as shown in the following pictures.







This defines a map $\#\colon \mathcal{K}not \times \mathcal{K}not \longrightarrow \mathcal{K}not$ called the *concatenation operation*. Let $\#_*$ denote the induced map in homology, and consider the following composite

$$H_*(\mathcal{K}not) \otimes H_*(\mathcal{K}not) \longrightarrow H_*(\mathcal{K}not \times \mathcal{K}not) \xrightarrow{\#_*} H_*(\mathcal{K}not)$$

in which the first map is the Künneth morphism. Let us denote this composite by $\times_1$. We thus have a morphism

$$\times_1\colon H_*(\mathcal{K}not) \otimes H_*(\mathcal{K}not) \longrightarrow H_*(\mathcal{K}not)$$

which endows the homology $H_*(\mathcal{K}not)$ with the structure of a commutative algebra.

There is a second, more subtle, operation on the homology $H_*(\mathcal{K}not)$ of the space of long knots which is induced by a map

$$\phi\colon S^1 \times \mathcal{K}not \times \mathcal{K}not \longrightarrow \mathcal{K}not,$$

where S^1 is the unit circle. For two given long knots $f, g \in \mathcal{K}not$ the map

$$\phi(-, f, g)\colon S^1 \longrightarrow \mathcal{K}not,\ z \mapsto \phi(z, f, g)$$

can be geometrically understood as making the knot f pass through the knot g and the knot g pass through the knot f, as is shown in the picture below.

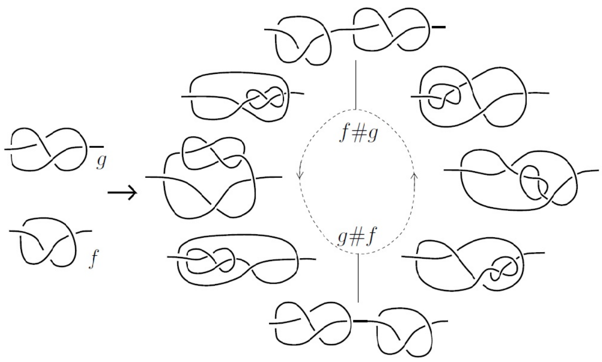
Passing to homology, the map ϕ induces a map

$$H_*(S^1) \otimes H_*(\mathcal{K}not) \otimes H_*(\mathcal{K}not) \longrightarrow H_*(\mathcal{K}not),$$

and restricting to the generator of $H_1(S^1)$ we get an operation

$$\{-,-\}_1\colon H_p(\mathcal{K}not) \otimes H_q(\mathcal{K}not) \longrightarrow H_{p+q+1}(\mathcal{K}not).$$

Let $\overline{\mathcal{K}not}$ be the following variation of the space of long knots. An element of $\overline{\mathcal{K}not}$ can be viewed as a thickened long knot. As before, there is also a

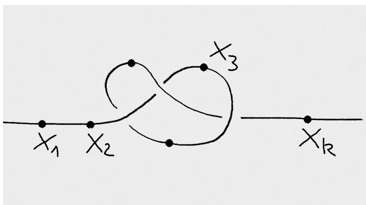


product $\times_1$ and a bracket $\{-,-\}_1$ on the homology $H_*(\overline{\mathcal{K}not})$, which turns out to be a Gerstenhaber algebra as proven by Budney in [8]. There is another Gerstenhaber algebra structure [1] (constructed by Sinha [44] and McClure-Smith [32]) that we would like to understand. The first result in this thesis will be a full explicitation of this latter Gerstenhaber algebra structure.

The study of the space of long knots is based on a construction of Sinha inspired by the Goodwillie-Weiss calculus [19]. Intuitively, a long knot can be discretized as a sequence of distinct points (see picture below), and this leads to consider the configuration spaces in the study of $\mathcal{K}not$. Recall that the *configuration space* of k points in $\mathbb{R}^d$ is the space

$$\mathrm{Conf}(k,\mathbb{R}^d) = \{(x_1,\cdots,x_k) \in (\mathbb{R}^d)^k |\ x_i \neq x_j \text{ whenever } i \neq j\}.$$

There exists compactifications $\mathrm{Conf}\left\langle k,\mathbb{R}^d\right\rangle$ weakly equivalent to $\mathrm{Conf}(k,\mathbb{R}^d)$. The spaces $\mathrm{Conf}\left\langle k,\mathbb{R}^d\right\rangle$ are very easy to understand, and constitute the building blocks of Sinha's cosimplicial model for the space $\overline{\mathcal{K}not}$ (a *cosimplicial model* can be understood as a combinatorial model). More precisely Sinha shows [44]



that there is a cosimplicial space (known as *Sinha's cosimplicial space*)

$$\mathrm{Conf}\left\langle 0,\mathbb{R}^d\right\rangle \rightleftarrows \mathrm{Conf}\left\langle 1,\mathbb{R}^d\right\rangle \rightleftarrows \mathrm{Conf}\left\langle 2,\mathbb{R}^d\right\rangle \cdots$$

[1] It is the Gerstenhaber algebra structure induced by the action of an E_2 operad built by McClure and Smith in [32]. The question to know whether the McClure-Smith action and the Budney action are equivalent is still open in my knowledge

whose totalization or total space is weakly equivalent to $\overline{\mathcal{K}not}$. The advantage of this result is one can compute the homology $H_*(\overline{\mathcal{K}not})$, based on the fact that the homology $H_*(\mathrm{Conf}\left\langle k, \mathbb{R}^d\right\rangle)$ is well known (see Fred Cohen [10]), by using a powerful tool known as *Bousfield-Kan spectral sequence*. Recall that a *spectral sequence* is a tool for computing some algebraic objects, in particular homology, by taking "successive approximations". It can be viewed as a book (with an infinite number of pages!) where the homology of one page gives the following page. The page ∞ encodes the "useful information" for computations. Sometimes, from some page, we read exactly the same thing in the rest of the book. In that case we say that the spectral sequence *collapses* at that page, which therefore contains all the "useful information". So it is very interesting when a spectral sequence collapses because we can then make computations easily. In [26], Lambrechts, Turchin and Volić proves that for $d \geq 4$ the spectral sequence computing the homology of $\overline{\mathcal{K}not}$ collapses at the E^2 page rationally. This E^2 page was extensively studied by several authors, and one of the first results is the fact that it is isomorphic to the graded vector space $\bigoplus\limits_{k\geq 0} H_*(\mathrm{Conf}\left\langle k, \mathbb{R}^d\right\rangle)$. It is also equiped with the natural structure of a Gerstenhaber algebra that we recall now. First define a map

$$\mathrm{Conf}\left\langle 2, \mathbb{R}^d\right\rangle \times \mathrm{Conf}\left\langle p, \mathbb{R}^d\right\rangle \times \mathrm{Conf}\left\langle q, \mathbb{R}^d\right\rangle \longrightarrow \mathrm{Conf}\left\langle p+q; \mathbb{R}^d\right\rangle$$

that sends $a = (a_1, a_2) \in \mathrm{Conf}\left\langle 2, \mathbb{R}^d\right\rangle$, $x \in \mathrm{Conf}\left\langle p, \mathbb{R}^d\right\rangle$ and $y \in \mathrm{Conf}\left\langle q, \mathbb{R}^d\right\rangle$ to the configuration of $p+q$ points obtained by "replacing" a_1 and a_2 respectively by x and y. The induced map in homology gives a map

$$\beta\colon H_0(S^{d-1})\otimes H_r(\mathrm{Conf}\left\langle p, \mathbb{R}^d\right\rangle)\otimes H_s(\mathrm{Conf}\left\langle q, \mathbb{R}^d\right\rangle) \longrightarrow H_{r+s}(\mathrm{Conf}\left\langle p+q; \mathbb{R}^d\right\rangle),$$

which defines a product $\times_2 = \beta(\mu \otimes -)$. Here μ is the generator of $H_0(S^{d-1})$. Next define maps

$$\circ_i\colon \mathrm{Conf}\left\langle p, \mathbb{R}^d\right\rangle \times \mathrm{Conf}\left\langle q, \mathbb{R}^d\right\rangle \longrightarrow \mathrm{Conf}\left\langle p+q-1, \mathbb{R}^d\right\rangle,\ 1 \leq i \leq p,$$

that send x and y to the configuration of $p+q-1$ points obtained by "replacing" the ith point x_i of x by the configuration y. Let us denote again by $\circ_i$ the induced map in homology, and define a bracket $\{-,-\}_2$ by $\{x,y\}_2 = \pm\sum_{i=1}^{p} x \circ_i y \pm \sum_{j=1}^{q} y \circ_j x$. For appropriate signs, see formula (3.2.6) from Chapter 3. Notice that the product $\times_2$ can also be expressed in term of the insertion map $\circ_i$: $x \times_2 y = (\mu \circ_2 y) \circ_1 x$. Define finally a differential $D_{\mathcal{K}}$ by $D_{\mathcal{K}}(x) = \{\mu, x\}_2$. Now equip the vector space $\bigoplus\limits_{k\geq 0} (H_*\mathrm{Conf}\left\langle k, \mathbb{R}^d\right\rangle)$ with the differential $D_{\mathcal{K}}$. It is a chain complex, and its homology, equipped with the product $\times_2$ and the bracket $\{-,-\}_2$, turns out to be a Gerstenhaber algebra.

From now on we have two Gerstenhaber algebras: the homology of the space of long knots endowed with the Gerstenhaber algebra structure (constructed by Sinha and McClure-Smith) we said before, and the explicit one we just defined in term of the homology of configuration spaces. At first glance, they appear quite different, but, somewhat miraculously, they turn out to be related. More precisely, we have the following theorem.

Theorem A. (Theorem 4.1.5) *Let the real numbers be the set of coefficients. Then, for $d \geq 4$, there is an isomorphism*

$$H_*(\overline{\mathcal{K}not}) \cong H_*\left(\bigoplus_{k\geq 0}(H_*\mathit{Conf}\left\langle k, \mathbb{R}^d\right\rangle), D_{\mathcal{K}}\right) \tag{0.0.1}$$

that respects the Gerstenhaber algebra structure.

Notice that it was already proved in [26] that both sides of (0.0.1) where isomorphic as *graded vector spaces*. Notice also that Cattaneo-Cotta-Ramusino-Longoni [9] and Turchin [54] compute some homology classes of the space of long knots. Our Theorem A enables us to determine if certain homology classes are decomposable or not. On that point, we furnish a table at the end of Chapter 4.

In order to prove Theorem A we need to introduce the notion of *operad* (nonsymmetric). Roughly speaking, an element of an operad is an operation with many inputs (their number is called the *arity*) and one output. An element of the structure of an operad is a map that sends two operations of arity p and q to an operation of arity $p + q - 1$. The example of operad we look at is the collection

$$\mathrm{Conf}\left\langle \bullet, \mathbb{R}^d\right\rangle = \{\mathrm{Conf}\left\langle k, \mathbb{R}^d\right\rangle\}_{k\geq 0},$$

equipped with maps $\circ_i$ defined before. This operad is known in the litterature as *Kontsevich's operad.* When $d = 1$, we get the *nonsymmetric operad* $\mathrm{Conf}\langle \bullet, \mathbb{R}\rangle$, also called the *associative operad*, which is very simple. If there is a morphism from the associative operad to another operad $\mathcal{O}(\bullet)$, we say that $\mathcal{O}(\bullet)$ is a *multiplicative operad.* For instance the standard linear inclusion $\mathbb{R} \hookrightarrow \mathbb{R}^d$ induces a morphism $\mathrm{Conf}\langle \bullet, \mathbb{R}\rangle \longrightarrow \mathrm{Conf}\left\langle \bullet, \mathbb{R}^d\right\rangle$, and therefore the Kontsevich operad $\mathrm{Conf}\left\langle \bullet, \mathbb{R}^d\right\rangle$ is multiplicative.

Maxim Kontsevich introduced the notion of *formality* of an operad in his famous paper [25]. Let us be precise about what this means. A morphism between two chain complexes is said to be a *quasi-isomorphism* (we will denote it by $\xrightarrow{\sim}$) if it induces an isomorphism in homology. An operad $\mathcal{O}$ is said to be *formal* if its singular chain complex $S_*\mathcal{O}$ and its homology $H_*\mathcal{O}$ are connected by a zigzag of quasi-isomorphisms of operads. In 1999, Kontsevich [25] proved that $\mathrm{Conf}\left\langle \bullet, \mathbb{R}^d\right\rangle$ is formal. Ten years later, P. Lambrechts and I. Volić [27] developed the details of Kontsevich's proof, and construct in particular an explicit zigzag

$$S_*\mathrm{Conf}\left\langle \bullet, \mathbb{R}^d\right\rangle \xleftarrow{\sim} \cdots \xrightarrow{\sim} H_*\mathrm{Conf}\left\langle \bullet, \mathbb{R}^d\right\rangle.$$

Since the operads $S_*\mathrm{Conf}\left\langle \bullet, \mathbb{R}^d\right\rangle$ and $H_*\mathrm{Conf}\left\langle \bullet, \mathbb{R}^d\right\rangle$ are all multiplicative, it is natural to ask whether they are connected by a zigzag of multiplicative operads. The following theorem answers this question.

Theorem B. (Theorem 4.1.1) *For $d \geq 3$ the Kontsevich operad is formal over reals as a multiplicative operad*

Notice that in [27] it is only proved that the operad $\mathrm{Conf}\left\langle \bullet, \mathbb{R}^d\right\rangle$ is formal as "up to homotopy multiplicative operad". Theorem B admits an interesting corollary in the sense that it simplifies considerably the main result of [26] (or Theorem 2.4.1). Recall that the notion of formality is defined in a similar way for cosimplicial spaces. That is, a cosimplicial space is formal if its singular chain complex can be "replaced" by its homology.

Corollary C. (Corollary 4.1.3) *For $d \geq 3$, Sinha's cosimplicial space is formal over reals.*

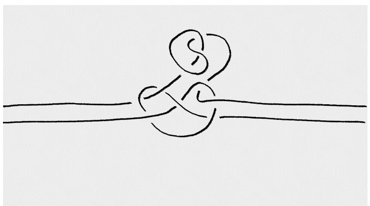
Notice that the main result of [26] is proved only for $d > 3$, but our approach also does the work in the three-dimensional space (as shown by Corollary C).

It is known that there exists *cosimplicial machinery* (built by McClure and Smith in [32]) that takes a multiplicative operad as input and produces a cosimplicial space whose totalization admits an action of an operad $\mathcal{D}_2$ weakly equivalent to $\mathrm{Conf}\langle\bullet, \mathbb{R}^2\rangle$. This operad $\mathcal{D}_2$ has a complicated description. In [41] Salvatore shows that it is isomorphic to a much simpler operad: the *cacti operad* (an operation of arity k is a cactus with k lobes). This latter operad has a nice geometric description, and acts (see Salvatore again in [41]) on the totalization. The natural question one can ask is whether the McClure-Smith action and the cacti action are equivalent. The following theorem gives a positive answer.

Theorem D. (Theorem 3.3.19) *Let $\mathcal{O}^\bullet$ be a cosimplicial space associated to a multiplicative operad. Then the McClure-Smith and the cacti operad actions on the totalization of $\mathcal{O}^\bullet$ are equivalent.*

To prove Theorem D, we will explicitly construct (in a more combinatorial way) the cacti operad action.

At the beginning of the introduction, we mentioned that we will study generalizations of long knots spaces. As first generalization, we have the *space of long links*. Let us precise what it means. A *long link* of m strands is a smooth embedding $f\colon \coprod_{i=1}^m \mathbb{R} \hookrightarrow \mathbb{R}^d$ of m copies of $\mathbb{R}$ inside the Euclidean space $\mathbb{R}^d$ that coincides outside a compact set with a fixed linear embedding (see Definition 5.1.1). A long link of one strand is a *long knot*. The following picture is that of a long link of 2 strands.



The *space of long links of m strands* , denoted by $\mathcal{L}ink_m$, is defined to be the collection of all long links of m strands equipped with a suitable topology (see Definition 5.1.2). The homology $H_*(\mathcal{L}ink_m)$ is equipped with the structure of an algebra, the product being induced by the concatenation operation as in the case of long knots. The lack of a bracket comes essentially from the fact that it is not possible in general to "pull one long link through another".

The study of the space $\mathcal{L}ink_m$ is mainly based on the theory of Goodwillie-Weiss [19], which allows rebuilding the space of embeddings of one manifold inside another by means of configuration spaces of points in the target manifold. By this theory, an element of the configuration space $\mathrm{Conf}\left\langle mk, \mathbb{R}^d\right\rangle$ of mk points in $\mathbb{R}^d$ should be thought as a configuration of $m \times k$ points such that there are exactly k points on each strand of a long link. From this point of view, there is a map

$$d^i \colon \mathrm{Conf}\left\langle mk, \mathbb{R}^d\right\rangle \longrightarrow \mathrm{Conf}\left\langle m(k+1), \mathbb{R}^d\right\rangle$$

sending a configuration of mk points to the configuration of $m(k+1)$ points obtained by "doubling" simultaneously the ith point of each strand. Let us denote again by d^i the induced map in homology, and define a map $D_{\mathcal{L}}$ as the alternating sum of d^i's, that is, $D_{\mathcal{L}} = \sum_i (-1)^i d^i$. Since the homology of configuration spaces is explicitly computable, and since the maps d^i are defined explicitly, it follows that the map $D_{\mathcal{L}}$ is explicit. It turns out to be a differential such that the couple $(\bigoplus\limits_{k\geq 0} H_*(\mathrm{Conf}\left\langle mk, \mathbb{R}^d\right\rangle), D_{\mathcal{L}})$ is a chain complex. The homology of that chain complex is therefore explicitly computable, and is related to the homology of $\mathcal{L}ink_m$. To be precise, we have the following theorem in which $\overline{\mathcal{L}ink}_m$ is a variation of the space of long links.

Theorem E. (Theorem 5.1.3 and Corollary 5.1.4) *Let the rational numbers be the set of coefficients. Then there is an isomorphism of vector spaces*

$$H_*(\overline{\mathcal{L}ink}_m) \cong H_*\left(\bigoplus_{k\geq 0}\left(H_*(\mathit{Conf}\left\langle mk, \mathbb{R}^d\right\rangle), D_{\mathcal{L}}\right)\right).$$

This theorem with $m = 1$ looks like Theorem A if one considers only the vector space structures. But the methods of proof are completely different. This is due to the fact that the space of long knots can be rebuilt from an operad (the Kontsevich operad) while the space of long links cannot. To prove Theorem E we use a variation of the Goodwillie-Weiss theory developed by Arone and Turchin in [2].

Theorem E can be reformulated as follows. In [35] Munson and Volić construct in the same spirit as Sinha a cosimplicial space $\{\mathrm{Conf}\left\langle mk, \mathbb{R}^d\right\rangle\}_{k\geq 0}$ that gives a cosimplicial model for the space of long links. They also conjecture that the spectral sequence computing the homology $H_*(\mathcal{L}ink_m)$ collapses. The following result proves that conjecture.

Theorem F. (Theorem 5.1.6) *The spectral sequence associated to the Munson-Volić cosimplicial model for the space $\overline{\mathcal{L}ink}_m$ collapses at the E^2 page rationally.*

Theorem E and Theorem F allow us to make computations. Among other things, they allow us to prove the growth of the *Betti numbers* of the space $\overline{\mathcal{L}ink}_m$. More precisely, we can prove the following theorem.

Theorem G. (Theorem 5.1.8) *The radius of convergence of the Poincaré series for the space $\mathcal{L}ink_m$ is less than or equal to $(\frac{1}{m})^{\frac{1}{d-1}}$. Therefore the Betti numbers of $\mathcal{L}ink_m$ have at least an exponential growth.*

An immediate consequence of Theorem G is the following corollary.

Corollary H. (Corollary 5.1.9) *The radius of convergence of the Poincaré series for* $\mathcal{L}ink_m$ *tends to* 0 *as* m *goes to* ∞.

In the particular case $m = 1$, we have $\mathcal{L}ink_1 = \mathcal{K}not$, and Turchin [53] proves that the radius of convergence of the space $\mathcal{K}not$ is less that or equal to $(\frac{1}{\sqrt{2}})^{\frac{1}{d-1}}$. Hence the bound of Turchin is much better than our bound in Theorem G which is 1. Since it is easy to see that the space $(\mathcal{K}not)^{\times_m}$ of m copies of long knots is a retract (up to homotopy) of $\mathcal{L}ink_m$, it follows that the radius of convergence of $\mathcal{L}ink_m$ is also less than or equal to $(\frac{1}{\sqrt{2}})^{\frac{1}{d-1}}$. Our Corollary H furnishes a better upper bound for m large.

We have extended our study to the high dimensional analogue of the space of long links, that is, the space of smooth embeddings of m copies of $\mathbb{R}^n$ inside $\mathbb{R}^d$ that coincides outside of compact set with a fixed linear embedding. Let us denote it by $\mathcal{L}ink_m^n$. As before let $\overline{\mathcal{L}ink}_m^n$ be a variation of the space $\mathcal{L}ink_m^n$. Let us also denote by k_n the number of k-simplices minus 1 in the simplicial model of the wedge of m copies of the n dimensional sphere (this simplicial model is well known). The following theorem is a generalization of Theorem E.

Theorem I. (Theorem 5.5.2 and Corollary 5.5.3) *With rational coefficients, there is an isomorphism of vector spaces*

$$H_*(\overline{\mathcal{L}ink}_m^n) \cong H_*\left(\bigoplus_{k\geq 0} H_*(Conf\langle k_n, \mathbb{R}^d\rangle)\right).$$

Some of the results of this thesis appear already in our paper [47] (Theorem A, Theorem B), and in our preprint [49] (Theorem E, Theorem F, Theorem G).

Organization of the work

This work is divided into five chapters each of them starting with a detailed introduction. Here is an overview of these introductions.

- In Chapter 1 we recall some basic notions such as simplicial and cosimplicial objects, model categories, operads, and spectral sequences. We also prove that the *relative properness axiom*, which says that the pushout in the model or semimodel category of nonsymmetric operads of a weak equivalence along a cofibration gives a weak equivalence, holds in the semimodel category of nonsymmetric operads. This axiom will be used to prove Theorem B.

- In Chapter 2 we first recall the space of long knots. Next we review the Kontsevich operad, the Fulton-MacPherson operad, and the operad of *admissible diagrams*. These three operads will be used in Chapter 4 in proving Theorem B. We also review the Sinha cosimplicial model for the space of long knots by recalling the classical theory of Goodwillie-Weiss.

We will see a variation of that theory in the last chapter. Finally we recall the notion of formality of topological operads. Nothing is new in this chapter apart from some proofs of well known results.

- In Chapter 3 we first recall the definition of a Gerstenhaber algebra and of the Hochschild homology associated to a multiplicative operad in graded vector spaces. Next we recall the cacti operad and we explicitly construct its action on $\mathrm{Tot}\mathcal{O}^{\bullet}$ (many examples are given here). We end the chapter with the proof of Theorem D.

- In Chapter 4 we first prove the crucial Lemma 4.2.3. The key ingredient in proving this lemma is the relative properness axiom. Next we prove Theorem B (by using Lemma 4.2.3) and Corollary C. We also give a very short proof of the main result of [26] (or Theorem 2.4.1) with $d \geq 3$. The chapter ends with the proof of Theorem A followed by some computations.

- In Chapter 5 we first recall the space of long links. We also recall the fundamental notion of infinitesimal bimodules, and a version of Goodwillie-Weiss theory. Using this version, we prove Theorem E, Theorem F and Theorem I. From Theorem F and the main result of [24], we deduce Theorem G. We end the chapter with the Poincaré series for the space of long links modulo m copies of long knots.

CHAPTER 1

Basic Notions

1.1 Introduction

This chapter recalls the notions of simplicial and cosimplicial objects, model categories, operads and spectral sequences that we will need in this thesis. The only new thing here is Proposition 1.4.6.

Outline of the chapter

- In Section 1.2 we explain in detail the construction of a simplicial model for the wedge of m copies of the circle, which will be used in proving Theorem 5.1.6. We also recall the definition of the totalization and the homotopy totalization of a cosimplicial space.

- In Section 1.3 we recall axioms of model categories and study the example of chain complexes, which will be used in the proof of Theorem 4.1.1. We also recall the notion of semi-model categories.

- In Section 1.4 we first prove Proposition 1.4.6, which says that the axiom of relative properness holds in the semi-model category of nonsymmetric operads in any cofibrantly generated symmetric monoidal model category. This axiom will help to prove the crucial Lemma 4.2.3. We also give a list of the most used operads in this thesis. We end this section with the fundamental construction that associates a cosimplicial object to any multiplicative operad.

- In Section 1.5 we give an example of a spectral sequence that collapses at the E^2 page, and which is such that the algebraic structure on E^2 is not the same as that of the abutment. We also recall the definition of the homology Bousfield-Kan spectral sequence associated to a cosimplicial space.

1.2 Simplicial and cosimplicial objects

Throughout this section, $\mathcal{C}$ is an arbitrary category. For more explanations about simplicial and cosimplicial objects, we refer the reader to [28], [16], [56].

1.2.1 Simplicial objects

Let Δ' denote the category of nonempty totally ordered finite sets and non decreasing maps as morphisms.

Definition 1.2.1. *A* simplicial object *in* $\mathcal{C}$ *consists of a contravariant functor* $X\colon \Delta' \longrightarrow \mathcal{C}$ *from* Δ' *to* $\mathcal{C}$*.*

Simplicial objects in $\mathcal{C}$ and natural transformations form a category that we denote by s$\mathcal{C}$. When $\mathcal{C} = \mathrm{Set}$, the category of sets, X is called a *simplicial set* . If $\mathcal{C} = \mathcal{A}b$, the category of abelian groups, X is called a *simplicial abelian group* , and so on and so forth. On can define a simplicial object in $\mathcal{C}$ as a covariant functor $X\colon \Delta^{op} \longrightarrow \mathcal{C}$ from the opposite category of Δ to $\mathcal{C}$. Here Δ is the subcategory of Δ' whose objects are sets on the form $[n] = \{0, \cdots, n\}$, naturally ordered. Notice that the functor $\Delta' \longrightarrow \Delta$ that sends a set S to $[\mathrm{Card}(S) - 1]$ is an equivalence of categories. Notice also that we will adopt the second definition of a simplicial object, which is the one most used in the literature, in this thesis. One of the reasons for which it is easier to work with the category Δ instead of Δ' is the fact that the collection of morphisms in Δ admits a base $\{d^i, s^j\}_{i,j}$ (in the sense that every morphism in Δ can be written as a composition of some d^i and s^j). This base captures the combinatorial structure of a simplicial object, and it is defined as follows. The morphism $d^i\colon [n] \longrightarrow [n+1], 0 \leq i \leq n+1$ is defined by

$$d^i(x) = \begin{cases} x & \text{if} \quad x \leq i-1 \\ x+1 & \text{if} \quad x \geq i. \end{cases} \tag{1.2.1}$$

Notice that d^i is an injective map such that i does not lie in its image. The second important morphism $s^j\colon [n+1] \longrightarrow [n], 1 \leq j \leq n+1$ is defined by

$$s^j(x) = \begin{cases} x & \text{if} \quad x < j-1 \\ j-1 & \text{if} \quad x = j-1 \text{ or } x = j \\ x-1 & \text{if} \quad x > j. \end{cases} \tag{1.2.2}$$

Notice also that s^j is a surjective map such that $s^j(j-1) = s^j(j)$.

For a simplicial object $X\colon \Delta^{op} \longrightarrow \mathcal{C}$, we write X_n for $X([n])$, the image of $[n]$ under X. We also write $d_i\colon X_{n+1} \longrightarrow X_n$ for the image $X((d^i)^{op})$ of the opposite morphism $(d^i)^{op}$ under X, and $s_j\colon X_n \longrightarrow X_{n+1}$ for the image $X((s^j)^{op})$. The morphism d_i is called the *face morphism* or the *face map*, and s_j is called the *degeneracy morphism* or the *degeneracy map*. It is straightforward to check that morphisms d_i and s_j satisfy identities (1.2.3) below, which are called *simplicial relations*.

$$\boxed{\begin{array}{ll} 1.\ d_i d_j = d_{j-1} d_i \text{ if } i < j & 2.\ d_i s_j = s_{j-1} d_i \text{ if } i < j \\ 3.\ d_j s_j = id = d_{j+1} s_j & 4.\ d_i s_j = s_j d_{i-1} \text{ if } i > j+1 \\ 5.\ s_i s_j = s_{j+1} s_i \text{ if } i \leq j & \end{array}} \tag{1.2.3}$$

One can also define a simplicial object in $\mathcal{C}$ as a sequence $X_\bullet = \{X_n\}_{n\geq 0}$ of objects in $\mathcal{C}$ equipped with degeneracy morphisms $s_j \colon X_n \longrightarrow X_{n+1}$ and face morphisms $d_i \colon X_{n+1} \longrightarrow X_n$ satisfying simplicial relations (1.2.3).

Now assume that $X_\bullet \colon \Delta^{op} \longrightarrow \mathrm{sSet}$ is a simplcial object in sets. We want to define its geometric realization $|X_\bullet|$. Let us first define the *standard geometric k-simplex* Δ^k . For us, it is defined by

$$\Delta^k = \begin{cases} * & \text{if } \ k = 0 \\ \{(t_1, \cdots, t_k) \in [-1,1]^k : -1 \leq t_1 \leq \cdots \leq t_k \leq 1\} & \text{if } \ k \geq 1. \end{cases} \tag{1.2.4}$$

Throughout this thesis the standard simplex will be viewed as in (1.2.4). Define now two maps $d^i \colon \Delta^k \longrightarrow \Delta^{k+1}$ and $s^j \colon \Delta^{k+1} \longrightarrow \Delta^k$ as follows.

- For $0 \leq i \leq k+1$, for $t = (t_1, \cdots, t_k) \in \Delta^k$,

$$d^i(t) = \begin{cases} (-1, t_1, \cdots, t_k) & \text{if } i = 0 \\ (t_1, \cdots, t_i, t_i, \cdots, t_k) & \text{if } 1 \leq i \leq k \\ (t_1, \cdots, t_k, 1) & \text{if } i = k+1. \end{cases} \tag{1.2.5}$$

- For $1 \leq j \leq k+1, t = (t_1, \cdots, t_{k+1}) \in \Delta^{k+1}$,

$$s^j(t) = (t_1, \cdots, t_{j-1}, t_{j+1}, \cdots, t_{k+1}). \tag{1.2.6}$$

The *geometric realization* of $X_\bullet$ is then defined by

$$|X_\bullet| = (\coprod_{n\geq 0} X_n \times \Delta^n)/ \sim,$$

where the equivalence relation is generated by

$$(s_j(x), t) \sim (x, s^j(t)) \qquad \text{and} \qquad (x, d^i(t)) \sim (d_i(x), t).$$

This definition gives rise to a functor $|-| \colon \mathrm{sSet} \longrightarrow \mathrm{Top}$ from simplicial sets to topological spaces. The following example is actually the motivating example of simplicial sets.

Example 1.2.2. *Let X be a topological space. For $n \geq 0$, define $\mathrm{Sing}_n(X) = \{\sigma \colon \Delta^n \longrightarrow X |\ \sigma$ is continuous$\}$. Then the collection*

$$\mathrm{Sing}_\bullet(X) = \{\mathrm{Sing}_n(X)\}_{n\geq 0},$$

equipped with morphisms induced by (1.2.5) and (1.2.6), turns out to be a simplicial set called the simplicial set of singular simplices .

Example 1.2.3. *Let $n \geq 0$ be an integer. Then the simplicial set $\Delta^k_\bullet = \{\mathrm{Hom}_\Delta([n],[k])\}_{n\geq 0}$ (faces and degeneracies are induced by morphisms from (1.2.1) and (1.2.2)) is a simplicial model for the standard k-simplex Δ^k, that is $|\Delta^k_\bullet| \cong \Delta^k$.*

Let us see another example of a simplcial set, which will be used later in Chapter 5. Let $\Delta^1_\bullet$ as in the previous example, and let $\partial\Delta^1_\bullet$ denote its boundary. Define the simplicial set $S^1_\bullet$ to be the quotient

$$S^1_\bullet = \frac{\Delta^1_\bullet}{\partial\Delta^1_\bullet}.$$

It is clear that $S^1_\bullet$ is a simplicial model of the circle S^1. By looking at Δ^1_p as a nondecreasing sequence of length $p+1$ on the alphabet $\{0,\, 1\}$, each S^1_p is a finite set pointed at

$$* = \underbrace{0\cdots 0}_{p+1} \sim \underbrace{1\cdots 1}_{p+1},$$

and faces and degeneracies preserve this base point.
Define now the simplicial set $(\vee_{i=1}^m S^1)_\bullet$ to be the wedge of m copies of the simplicial set $S^1_\bullet$,

$$(\vee_{i=1}^m S^1)_\bullet = \vee_{i=1}^m (S^1_\bullet).$$

The following proposition is well known in the litterature.

Proposition 1.2.4. *The simplicial set $(\vee_{i=1}^m S^1)_\bullet$ is a simplicial model for the wedge $\vee_{i=1}^m S^1$. Moreover, for each $p \geq 0$, the finite pointed set $(\vee_{i=1}^m S^1)_p$ is of cardinal $mp+1$. That is,*

$$\mathrm{Card}((\vee_{i=1}^m S^1)_p) = mp+1. \tag{1.2.7}$$

Proof. It is straightforward to check that $(\vee_{i=1}^m S^1)_\bullet$ is a simplicial model of $\vee_{i=1}^m S^1$.
Let $p \geq 0$. Since $\mathrm{Card}(S^1_p) = p+1$, by the definition of the wedge, we have the equation (1.2.7).

□

1.2.2 Cosimplicial objects

A cosimplicial object in $\mathcal{C}$ is just the dual of a simplicial object in $\mathcal{C}$. More precisely, we have the following definition.

Definition 1.2.5. *A* cosimplicial object *in $\mathcal{C}$ is a covariant functor $Y\colon \Delta' \longrightarrow \mathcal{C}$ from Δ' to $\mathcal{C}$.*

When $\mathcal{C} = \mathrm{Top}$, we say that Y is a *cosimplicial space* , when $\mathcal{C} = \mathrm{sSet}$, Y is called a *cosimplicial simplicial set*, and so on and so forth. Cosimplcial objects and natural transformations form a category that we denote by c$\mathcal{C}$.

Remark 1.2.6. *Since the categories Δ and Δ' are equivalent in the traditional language of categories, a cosimplicial object in $\mathcal{C}$ can be viewed as a covariant functor $Y\colon \Delta \longrightarrow \mathcal{C}$ from Δ to $\mathcal{C}$. These two definitions of a cosimplicial object will be used interchangeably in this thesis. For instance, in Section 3.3 from Chapter 3, we will write Δ for Δ'.*

For a cosimplicial object $Y\colon \Delta \longrightarrow \mathcal{C}$, let Y^n denote the image $Y([n])$. Let $d^i\colon Y^n \longrightarrow Y^{n+1}$ denote the image of the morphism d^i from (1.2.1) under Y, and let $s^j\colon Y^{n+1} \longrightarrow Y^n$ denote the image of the morphism s^j from (1.2.2) under Y again. One can define a cosimplicial object as a sequence $Y^\bullet = \{Y^n\}_{n\geq 0}$ of objects in $\mathcal{C}$ equipped with morphisms d^i (called *coface morphism*) and s^j (called *codegeneracy morphism*) that satisfy the following identities, called *the cosimplicial relations* .

$$\boxed{\begin{array}{ll} 1.\ d^jd^i = d^id^{j-1} \text{ if } i<j & 2.\ s^jd^i = d^is^{j-1} \text{ if } i<j \\ 3.\ s^jd^j = id = s^jd^{j+1} & 4.\ s^jd^i = d^{i-1}s^j \text{ if } i>j+1 \\ 5.\ s^js^i = s^is^{j+1} \text{ if } i\leq j. & \end{array}} \tag{1.2.8}$$

Example 1.2.7. *The sequence $\Delta^\bullet = \{\Delta^n\}_{n\geq 0}$ equipped with maps d^i and s^j defined by (1.2.5) and (1.2.6) forms a cosimplicial space usually called the* standard cosimplicial space .

Definition 1.2.8. *The* totalization *of a cosimplicial space $Y^\bullet$, denoted by* $\mathrm{Tot}Y^\bullet$*, is the space of natural maps from the standard cosimplicial space $\Delta^\bullet$ to $Y^\bullet$. That is,*

$$\mathrm{Tot}Y^\bullet = \mathrm{Nat}(\Delta^\bullet, Y^\bullet).$$

It is topologized as a subspace of the product $\prod_{k\geq 0}\mathrm{Map}(\Delta^k, Y^k)$.

Here are some examples of cosimplicial spaces and their totalizations.

Example 1.2.9. *The sequence $*^\bullet = \{*\}_{n\geq 0}$ of one point spaces equipped with trivial maps $d^i\colon * \longrightarrow *$ and $s^j\colon * \longrightarrow *$ is obviously a cosimplicial space. We can easily see that its totalization* $\mathrm{Tot}*^\bullet$ *is the one point space.*

Example 1.2.10. *Let $(X,*)$ be a pointed topological space, and let $X^{\times\bullet} = \{X^{\times n}\}_{n\geq 0}$ be a sequence of spaces defined by*

$$X^{\times n} = \begin{cases} * & \text{if } \ n=0 \\ \underbrace{X\times\cdots\times X}_{n} & \text{if } \ n\geq 1. \end{cases}$$

Define two maps $d^i\colon X^{\times n} \longrightarrow X^{\times n+1}$ and $s^j\colon X^{\times n+1} \longrightarrow X^{\times n}$ by

$$d^i(x_1,\cdots,x_n) = \begin{cases} (*,x_1,\cdots,x_n) & \text{if } \ i=0 \\ (x_1,\cdots,x_i,x_i,\cdots,x_n) & \text{if } \ 1\leq i\leq n \\ (x_1,\cdots,x_n,*) & \text{if } \ i=n+1, \end{cases}$$

and

$$s^j(x_1,\cdots,x_{n+1}) = (x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_{n+1}), 1\leq j\leq n+1.$$

It is straightforward to see that d^i and s^j satisfy cosimplicial relations (1.2.8). Hence, $X^{\times\bullet}$ is a cosimplicial space that gives a cosimplicial model for the loop space ΩX of X. More precisely, there is a homeomorphism $\mathrm{Tot}X^{\times\bullet} \xrightarrow{\cong} \Omega X$ that sends a natural transformation $\alpha = (\alpha_k)_{k\geq 0}$ to $\alpha_1\colon \Delta^1 \longrightarrow X$ (the map α_1 is a loop because $\alpha_1 d^0 = \alpha_1 d^1$ by naturality, and $X^{\times 0} = *$).

The construction that associates a topological space $\mathrm{Tot}Y^\bullet$ to any cosimplicial space $Y^\bullet$ gives rise to a covariant functor

$$\mathrm{Tot}\colon \mathrm{cTop} \longrightarrow \mathrm{Top}$$

from cosimplicial spaces to topological spaces. This functor is not a homotopy invariant (in the sense that two *levelwise weakly equivalent* cosimplicial spaces does not have in general weakly equivalent totalizations). To get the homotopy invariant one, called *the homotopy totalization*, let us first recall the construction that associates a space to any category $\mathcal{C}$. Let $(N\mathcal{C})_\bullet$ be the simplicial set defined by $(N\mathcal{C})_n = \mathrm{Hom}([n], \mathcal{C})$, the set of functors from $[n]$ to $\mathcal{C}$ (notice that $(N\mathcal{C})_n$ can be viewed as the set of n composable morphisms in $\mathcal{C}$). The faces and degeneracies in $(N\mathcal{C})_\bullet$ are induced in the obvious way by morphisms from (1.2.1) and (1.2.2). Let $B\mathcal{C}$ denote the geometric realization of $(N\mathcal{C})_\bullet$. Define now a covariant functor

$$B(\mathcal{C} \downarrow -)\colon \mathcal{C} \longrightarrow \mathrm{Top}$$

as follows. For an object a in $\mathcal{C}$, let $\mathcal{C} \downarrow a$ be the category whose objects are couple (c, f) in which c is an object of $\mathcal{C}$ and $f\colon c \longrightarrow a$ is a morphism in $\mathcal{C}$. A morphism from (c, f) to (c', f') consists of a morphism $g\colon c \longrightarrow c'$ in $\mathcal{C}$ such that $f = f'g$. The functor $B(\mathcal{C} \downarrow -)$ is then defined by

$$B(\mathcal{C} \downarrow -)(a) = B(\mathcal{C} \downarrow a).$$

By taking $\mathcal{C} = \Delta$ as input in the previous construction, we obtain a covariant functor $B(\Delta \downarrow -)\colon \Delta \longrightarrow \mathrm{Top}$ that we denote by $\widetilde{\Delta}^\bullet$.

Definition 1.2.11. *The* homotopy totalization *of a cosimplicial space $Y^\bullet$ is the space* $\mathrm{hoTot}Y^\bullet$ *of natural transformations from $\widetilde{\Delta}^\bullet$ to $Y^\bullet$. That is,*

$$\mathrm{hoTot}Y^\bullet = \mathrm{Nat}(\widetilde{\Delta}^\bullet, Y^\bullet).$$

Notice that Definition 1.2.11 is nothing other than the definition of the homotopy limit of $Y^\bullet$. More generally, if $F\colon I \longrightarrow \mathrm{Top}$ is a covariant functor from a small category I (that is, a category in which the collection of objects and morphisms is a set) to spaces, one defines the *homotopy limit* of F as the space of natural transformations from $B(I \downarrow -)$ to F. That is,

$$\underset{I}{\mathrm{holim}} F = \mathrm{Nat}(B(I \downarrow -), F).$$

One also defines the *homotopy colimit* of F as the space

$$\underset{I}{\mathrm{hocolim}} F = B(I^{op} \downarrow -) \otimes_I F = (\coprod_{i \in I} B(I^{op} \downarrow i) \times F(i))/\sim,$$

where the equivalence relation $\sim$ is generated by

$$(x, F(\alpha)(y)) \sim (B(I^{op} \downarrow -)(\alpha)(x), y)$$

for each morphism $\alpha\colon i \longrightarrow j$ in I, $x \in B(I^{op} \downarrow j)$ and $y \in F(i)$. Notice that the functor $B(I^{op} \downarrow -)$ from I to topological spaces is a contravariant functor. A good reference for homotopy limits and homotopy colimits is [22, Chapter 18-Chapter 19] or [7].

1.3 Model categories and semi-model categories

In this section we just give some basic notions of the theory of model categories, which is well developed in [23], [22]. We also recall some notions about semi-model categories [13].

1.3.1 Model categories

A *model category* is a place where it is possible to set up the basic machinery of homotopy theory. Here are axioms of such a category. Let $\mathcal{X}$ be a category. A *model category structure* on $\mathcal{X}$ consists of three classes of morphisms, called *weak equivalences*, *cofibrations* and *fibrations*, so that the following axioms hold:
M1 (*two-out-of-three axiom*): Let f and g be composable morphisms. If any two among f, g, and fg are weak equivalences, then so is the third.
M2 (*retract axiom*): Suppose f is a retract of g. If g is a weak equivalence (respectively a fibration, a cofibration), then so is f.
M3 (*lifting axioms*):

- The cofibrations have the left lifting property with respect to the acyclic fibrations (recall that an acyclic fibration is a fibration which is a weak equivalence)

- The fibrations have the right lifting property with respect to the acyclic cofibrations.

M4 (*factorization axioms*):

- Any morphism has a factorization $f = pi$, where p is a acyclic fibration and i is a cofibration.

- Any morphism has a factorization $f = pi$, where p is a fibration and i is a acyclic cofibration.

Definition 1.3.1. *A* model category *is a category with all (small) limits and colimits that satisfy axioms M1, M2, M3 and M4.*

The existence of all (small) limits and colimits implies that any model category has an initial object, denoted by $\emptyset$, and a terminal object, denoted by $*$. By convention, an object A in a model category $\mathcal{X}$ is *cofibrant* if the initial morphism $\emptyset \longrightarrow A$ is a cofibration, *fibrant* if the terminal morphism $A \longrightarrow *$ is a fibration.

Example 1.3.2. *Among all examples of model categories such as topological spaces, chain complexes, simplicial sets, differential graded commutive algebras,..., we will give here just the example of non-negatively graded chain complexes because it will be useful in the proof of Theorem 4.1.1. By [23, Theorem 2.3.11], the category of chain complexes* Ch_R *of modules over a ring* R *is a model category. Recall that a morphism* $f\colon A_* \longrightarrow B_*$ *of chain complexes is a*

- *weak equivalence if* $H_*(f)$ *is an isomorphism;*
- *fibration if* $f_n\colon A_n \longrightarrow B_n$ *is surjective for* $n \geq 1$*;*

- *cofibration if and only if for each $n \geq 0$, the map $f_n \colon A_n \longrightarrow B_n$ is an injection with projective cokernel.*

It follows that every object in that category is fibrant, and the cofibrant objects are chain complexes formed by projective modules. So if R is a field then every object of Ch_R is cofibrant since a module over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space and we know that every vector space is free.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a *symmetric monoidal category* (see [23] for a basic reference on symmetric monoidal categories) in which all small colimits and all small limits exist. We assume that the tensor product $\otimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ preserves all colimits in each variable separately. This means that the natural morphism

$$\mathrm{colim}_i(A_i \otimes B) \longrightarrow (\mathrm{colim}_i A_i) \otimes B,$$

is an isomorphism for all diagrams A_i and every fixed object $B \in \mathcal{C}$.
For a set S and an object $C \in \mathcal{C}$, define a tensor product $S \otimes C$ to be the coproduct over S of copies of the object C. That is,

$$S \otimes C = \coprod_{s \in S} C. \tag{1.3.1}$$

Definition 1.3.3. *The category $\mathcal{C}$ is called* symmetric monoidal model category *if it is equipped with a model structure such that the following axioms hold in $\mathcal{C}$.*
unit axiom: The unit object $\mathbf{1}$ *is cofibrant in $\mathcal{C}$.*
pushout-product axiom: The natural morphism

$$(i_*, j_*) : A \otimes D \oplus_{A \otimes C} B \otimes C \longrightarrow B \otimes D$$

induced by cofibrations $i : A \rightarrowtail B$ and $j : C \rightarrowtail D$ forms a cofibration, respectively an acyclic cofibration if i or j is also acyclic.

Definition 1.3.4. *A model category is said to be* cofibrantly generated *if it is equipped with a set of generating cofibrations $\mathcal{I}$, and a set of generating acyclic cofibrations $\mathcal{J}$, such that*

- *the fibrations are characterized by the right lifting property with respect to the acyclic generating cofibrations $j \in \mathcal{J}$;*
- *The acyclic fibrations are characterized by the right lifting property with respect to the generating cofibrations $i \in \mathcal{I}$.*

The importance of cofibrantly generated model categories comes from the definition of new model categories by adjunction from a cofibrantly generated model structure. For instance, in Section 1.4, we apply the construction of adjoint model structures to get a model (or semi-model) structure on the category of operads.

From now and in the rest of this chapter and in Chapter 4, we assume that $\mathcal{C}$ is a symmetric monoidal model category that is cofibrantly generated.

1.3.2 Semi-model categories

Roughly speaking, a *semi-model category* is a category which satisfies all axioms of model category , including the lifting axiom and the factorization axiom, but only for morphisms $f\colon A \longrightarrow B$ whose domain A is a cofibrant object. More precisely, we have the following axioms.

A *semi-model category structure* on a category $\mathcal{X}$ consists of classes of weak equivalences, cofibrations and fibrations so that axioms M1, M2 of model categories hold, but where the lifting axiom M3 and the factorization axiom M4 are replaced by the weaker requirements:
M3':

- The fibrations have the right lifting property with respect to the acyclic cofibrations $i\colon A \longrightarrow B$ whose domain A is cofibrant.
- The acyclic fibrations have the right lifting property with respect to the cofibrations $A \longrightarrow B$ whose domain A is cofibrant.

M4':

- Any morphism $f\colon A \longrightarrow B$ such that the domain A is cofibrant has a factorization $f = pi$, where i is a cofibration and p is a acyclic fibration.
- Any morphism $f\colon A \longrightarrow B$ such that the domain A is cofibrant has a factorization $f = pi$, where i is a acyclic cofibration and p is a fibration.

Definition 1.3.5. *A* semi-model category *is a category with all (small) limits and colimits that satisfy axioms M1, M2, M3' and M4'.*

Since the lifting axiom M3' and the factorization axiom M4' are not sufficient to imply that the initial object of $\mathcal{X}$ is cofibrant, a semi-model category is then assumed to satisfy the following axiom.
M0' (*initial object axiom*): The initial object of $\mathcal{X}$ is cofibrant.

Notice that in a semi-model category, the class of (acyclic) cofibrations is not fully characterized by the left lifting axiom M3', and similarly as regards the class of (acyclic) fibrations. The notion of a cofibrantly generated model category has a natural generalization in the context of semi-model categories.

1.4 Operads

Here we review the homotopy theory for operads, and we recall some useful results for us. We also give a lot of examples of operads which will be used in this thesis. In all this section the category $\mathcal{C}$ is as in the previous section (that is, $\mathcal{C}$ is a symmetric monoidal model category that is cofibrantly generated). This category will serve as the base category for operads. This section is based on [13].

1.4.1 The category of symmetric operads

We begin with the definition of a symmetric operad. A Σ_**-object* in $\mathcal{C}$ consists of a sequence $\{X_n\}_{n\in\mathbb{N}}$, where X_n is an object of $\mathcal{C}$ equipped with an action of the symmetric group Σ_n. We denote the category of Σ_*-objects in $\mathcal{C}$ by $\mathcal{M}$. This category turns to be a symmetric monoidal category equipped with a cofibrantly generated model structure defined in [13].

Definition 1.4.1. *A symmetric operad in $\mathcal{C}$ consits of a symmetric sequence $P = \{P(n)\}_{n\geq 0}$ equipped with an unit element $\mathbf{1} \longrightarrow P(1)$, which is just a morphism in $\mathcal{C}$, and a collection of morphisms*

$$\gamma\colon P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k) \longrightarrow P(i_1 + \cdots + i_k)$$

that satisfy natural equivariance properties, unit and associative axioms (May's axioms, see [29, Definition 1.1]*).*

Let $\mathcal{O}p_s(\mathcal{C})$ or $\mathcal{O}p_s$ denote the category of symmetric operads in $\mathcal{C}$. This category is in general endowed with only a semi-model structure defined as follows. The category $\mathcal{C}^{\mathbb{N}}$ of sequences of objects $X = \{X(n)\}_{n\geq 0}$ in $\mathcal{C}$ is endowed with a model structure induced by $\mathcal{C}$ levelwise. More explicitly, the weak equivalences, the cofibrations and fibrations in $\mathcal{C}^{\mathbb{N}}$ are all levelwise. Since the model structure in $\mathcal{C}$ is cofibrantly generated, then so is the model structure in $\mathcal{C}^{\mathbb{N}}$.
Let $U_2\colon \mathcal{M} \longrightarrow \mathcal{C}^{\mathbb{N}}$ be the obvious forgetful functor from $\mathcal{M}$ to $\mathcal{C}^{\mathbb{N}}$. It is clear that U_2 admits a left adjoint $F_2\colon \mathcal{C}^{\mathbb{N}} \longrightarrow \mathcal{M}$, which maps a collection X to the object $\Sigma_* \otimes X$ defined by $(\Sigma_* \otimes X)(n) = \Sigma_n \otimes X(n)$ (see (1.3.1) for the definition of a tensor product of a set by an object in $\mathcal{C}$).
Let $U_1\colon \mathcal{O}p_s \longrightarrow \mathcal{M}$ be the forgetful functor from $\mathcal{O}p_s$ to $\mathcal{M}$. It is also clear that this functor admits a left adjoint $F_1\colon \mathcal{M} \longrightarrow \mathcal{O}p_s$. We thus obtain the following pairs of functors.

$$F_1\colon \mathcal{M} \rightleftarrows \mathcal{O}p_s : U_1 \quad \text{and} \quad F_2\colon \mathcal{C}^{\mathbb{N}} \rightleftarrows \mathcal{M} : U_2.$$

In many other examples, we have a natural adjunction relation $F\colon \mathcal{Z} \rightleftarrows \mathcal{Y} : U$, where $\mathcal{Z}$ is a reference model category, and a model structure on $\mathcal{Y}$ is defined by that of $\mathcal{Z}$. This means that a morphism in $\mathcal{Y}$ is a weak equivalence (respectively a fibration) if its image under U yields a weak equivalence in $\mathcal{Z}$ (respectively a fibration in $\mathcal{Z}$). To see that this definition endows $\mathcal{Y}$ with a model structure, one must check the axioms M1, M2, M3, M4, and this is difficult in general.
However, in the context of cofibrantly generated model categories, the verifications can be reduced to simple conditions as we can see in [13, Theorem 11.1.13]. For instance, one can see that these simple conditions hold with the adjunction $F_2\colon \mathcal{C}^{\mathbb{N}} \rightleftarrows \mathcal{M} : U_2$, but do not hold with (F_1, U_1). So we can not apply the adjunction construction of model structures to get a model structure on the category $\mathcal{O}p_s$. However, if we restrict the lifting and the factorization axioms of model categories to morphisms with a cofibrant domain (in order words, if we work with semi-model categories), then these simple conditions are now verified by the pair $F_1\colon \mathcal{M} \rightleftarrows \mathcal{O}p_s : U_1$. Hence, the category $\mathcal{O}p_s$ is endowed with a semi-model structure [13, Theorem 12.2.A].

The following theorem says that the semi-model category $\mathcal{O}p_s$ of symmetric operads satisfies the axiom of relative properness.

Theorem 1.4.2. [13, Theorem 12.2.B] *Let P and Q be two symmetric operads that are Σ_*-cofibrant. Then the pushout of a weak equivalence along a cofibration*

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, "\sim"'] & R \arrow[d, dotted] \\ Q \arrow[r, dotted] & S \end{tikzcd}$$

gives a weak equivalence $R \xrightarrow{\sim} S$.

1.4.2 The category of nonsymmetric operads

The goal of this section is to state and prove Proposition 1.4.6, which is the nonsymmetric version of Theorem 1.4.2.

Roughly speaking, a *nonsymmetric operad* in $\mathcal{C}$ is a symmetric operad $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ in which we have omitted, for each $n \geq 0$, the Σ_n action on $\mathcal{O}(n)$. More explicitly, we have the following definition.

Definition 1.4.3. *A* nonsymmetric operad *in $\mathcal{C}$ is a collection $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ of objects in $\mathcal{C}$ together with an unit element $\eta\colon \mathbf{1} \longrightarrow \mathcal{O}(1)$ and the compostition or the* insertion map

$$\circ_i\colon \mathcal{O}(r) \otimes \mathcal{O}(s) \longrightarrow \mathcal{O}(r+s-1), 1 \leq i \leq r,$$

satisfying the following three axioms

(OP)$_1$: For $1 \leq i \leq r$ and $1 \leq j \leq s$, the following diagram commutes

$$\begin{tikzcd} \mathcal{O}(r) \otimes \mathcal{O}(s) \otimes \mathcal{O}(t) \arrow[r, "\circ_i \otimes id"] \arrow[d, "id\otimes\circ_j"'] & \mathcal{O}(r+s-1) \otimes \mathcal{O}(t) \arrow[d, "\circ_{i-1+j}"] \\ \mathcal{O}(r) \otimes \mathcal{O}(s+t-1) \arrow[r, "\circ_i"] & \mathcal{O}(r+s+t-2). \end{tikzcd}$$

(OP)$_2$: For $1 \leq j < i \leq r$, the following diagram commutes

$$\begin{tikzcd} \mathcal{O}(r) \otimes \mathcal{O}(s) \otimes \mathcal{O}(t) \arrow[r, "id\otimes T"] \arrow[d, "\circ_j \otimes id"'] & \mathcal{O}(r) \otimes \mathcal{O}(t) \otimes \mathcal{O}(s) \arrow[r, "\circ_i \otimes id"] & \mathcal{O}(r+t-1) \otimes \mathcal{O}(s) \arrow[d, "\circ_j"] \\ \mathcal{O}(r+s-1) \otimes \mathcal{O}(t) \arrow[rr, "\circ_{i-1+s}"] & & \mathcal{O}(r+s+t-2). \end{tikzcd}$$

Here $T\colon \mathcal{O}(s) \otimes \mathcal{O}(t) \xrightarrow{\cong} \mathcal{O}(t) \otimes \mathcal{O}(s)$ is the isomorphism coming from the symmetric structure of $\mathcal{C}$.

(OP)$_3$: For $1 \leq i \leq r$, the following two compositions are equal to the identity

$$\mathcal{O}(r) \xrightarrow{\cong} \mathbf{1} \otimes \mathcal{O}(r) \xrightarrow{\eta\otimes id} \mathcal{O}(1) \otimes \mathcal{O}(r) \xrightarrow{\circ_1} \mathcal{O}(r)\ ,$$

$$\mathcal{O}(r) \xrightarrow{\cong} \mathcal{O}(r) \otimes \mathbf{1} \xrightarrow{id\otimes\eta} \mathcal{O}(r) \otimes \mathcal{O}(1) \xrightarrow{\circ_i} \mathcal{O}(r)\ .$$

Definition 1.4.4. *A* morphism $f\colon \mathcal{O} \longrightarrow \mathcal{O}'$ *between two nonsymmetric operads consists of a collection* $\{f_n\colon \mathcal{O}(n) \longrightarrow \mathcal{O}'(n)\}_{n\geq 0}$ *of morphisms in* $\mathcal{C}$*, sending the unit of* $\mathcal{O}$ *to the one of* $\mathcal{O}'$*, such that the following square*

$$\begin{tikzcd}
\mathcal{O}(r)\otimes\mathcal{O}(s) \arrow[r, "\circ_i"] \arrow[d, "f_r\otimes f_s"'] & \mathcal{O}(r+s-1) \arrow[d, "f_{r+s-1}"'] \\
\mathcal{O}'(r)\otimes\mathcal{O}'(s) \arrow[r, "\circ_i'"] & \mathcal{O}'(r+s-1)
\end{tikzcd}$$

commutes for all $1 \leq i \leq r$*.*

Let $\mathcal{O}p_{ns}(\mathcal{C})$ or just $\mathcal{O}p_{ns}$ denote the category of nonsymmetric operads in $\mathcal{C}$. This category, as the category $\mathcal{O}p_s$, usually has only semi-model structure, by the nonsymmetric version of [13, Theorem 12.2.A]. But, if $\mathcal{C}$ satisfies the monoid axiom (see [36, Definition 6.1]), then the category $\mathcal{O}p_{ns}$ is equipped with a model structure cofibrantly generated (see [36, Theorem 1.1]) if we assume that the sets of generating cofibrations and generating acyclic cofibrations in $\mathcal{C}$ have presentable sources. Recall that an object A of $\mathcal{C}$ is *presentable* if there exists a cardinal λ such that the representable functor $\mathcal{C}(A,-)\colon \mathcal{C} \longrightarrow \mathrm{Set}$ commutes with λ-filtered colimits in $\mathcal{C}$. For instance, it is not difficult to see that the category of non-negatively graded chain complexes $\mathrm{Ch}_{\mathbb{R}}$ satisfies such conditions.

Remark 1.4.5. *The category* $\mathcal{O}p_{ns}(\mathrm{Ch}_{\mathbb{R}})$ *is equipped with a model category structure in which fibrations and weak equivalences are all levelwise.*

Let $U\colon \mathcal{O}p_s(\mathcal{C}) \longrightarrow \mathcal{O}p_{ns}(\mathcal{C})$ be the obvious forgetful functor. It is clear that this forgetful functor admits a left adjoint $\mathrm{Sym}\colon \mathcal{O}p_{ns}(\mathcal{C}) \longrightarrow \mathcal{O}p_s(\mathcal{C})$ defined by

$$\mathrm{Sym}(P)(n) = \Sigma_n \otimes P(n) = \coprod_{\sigma\in\Sigma_n} P(n).$$

The following proposition was first proved by Spitzweck [50] and by Berger-Moerdijk [3] for symmetric operads.

Proposition 1.4.6. *The category of nonsymmetric operads in* $\mathcal{C}$ *satisfies the axiom of relative properness : If* P *and* Q *are objects of* $\mathcal{O}p_{ns}$ *that are cofibrant in* $\mathcal{C}^{\mathbb{N}}$*, then the pushout in* $\mathcal{O}p_{ns}$ *of a weak equivalence along a cofibration*

$$\begin{tikzcd}
P \arrow[r, tail] \arrow[d, "\sim"'] & R \arrow[d, dotted] \\
Q \arrow[r, dotted] & S
\end{tikzcd}$$

gives a weak equivalence $R \xrightarrow{\sim} S$*.*

Proof. Since P and Q are cofibrant in the category $\mathcal{C}^{\mathbb{N}}$, it follows that $\mathrm{Sym}(P)$ and $\mathrm{Sym}(Q)$ are Σ_*-cofibrant. By applying the functor Sym to the diagram of the statement, we obtain the following pushout diagram in the category of symmetric operads

$$\begin{tikzcd} \mathrm{Sym}(P) \arrow[r, tail] \arrow[d, "\sim"'] & \mathrm{Sym}(R) \arrow[d, dotted] \\ \mathrm{Sym}(Q) \arrow[r, dotted] & \mathrm{Sym}(S) \end{tikzcd} .$$

We apply now [13, Theorem 12.2.B] to get a weak equivalence $\mathrm{Sym}(R) \xrightarrow{\sim} \mathrm{Sym}(S)$. Since $\mathrm{Sym}(R)(n)$ (respectively $\mathrm{Sym}(S)(n)$) is the coproduct over the set Σ_n of copies of the object $R(n)$ (respectively$S(n)$), it follows that the morphism $R \longrightarrow S$ in $\mathcal{O}p_{ns}$ is also a weak equivalence. $\square$

1.4.3 Some examples of operads

In this section and in next chapters, our operads are supposed to be nonsymmetric. Let us start with the most fundamental example of an operad.

Example 1.4.7. *Let X be an object in $\mathcal{C}$. The endomorphism operad End_X is the operad defined by*

$$End_X(n) = \hom_{\mathcal{C}}(X^n, X),$$

where X^n is the tensor product of n copies of X. The ith insertion morphism $\circ_i$ is defined as follows. For $f \in End_X(p)$ and $g \in End_X(q)$,

$$f \circ_i g = f \circ (\underbrace{id \otimes id}_{i-1} \otimes g \otimes \underbrace{id \otimes id}_{p-i}).$$

The unit $\mathbf{1} \in End_X(1)$ is the identity morphism $id\colon X \longrightarrow X$.

In next chapters we will see many actions of operads.

Definition 1.4.8. *Let $\mathcal{O}$ be an operad in $\mathcal{C}$, and let X be an object in $\mathcal{C}$. We will say that $\mathcal{O}$* acts *on X (or X is an $\mathcal{O}$-algebra) if there is a morphism $\mathcal{O} \longrightarrow End_X$ of operads.*

The following operad is just the dual of Example 1.4.7.

Example 1.4.9. *Let X be a topological space. The coendomorphism operad $Coend_X$ is the operad defined by $Coend_X(n) = Map(X, X^n)$. The ith insertion morphism $\circ_i$ is defined as follows. For*

$$f = (f_1, \cdots, f_p) \in Coend_X(p), \qquad g = (g_1, \cdots, g_q) \in Coend_X(q),$$

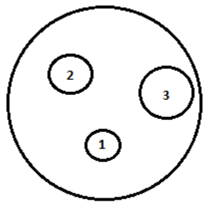
we have

$$f \circ_i g = (f_1, \cdots, f_{i-1}, g_1 \circ f_i, \cdots, g_q \circ f_i, f_{i+1}, \cdots, f_p)\colon X \longrightarrow X^{p+q-1}.$$

The unit $\mathbf{1} \in Coend_X(1)$ is the identity morphism $id\colon X \longrightarrow X$.

A powerful technique to study a manifold is to "scan" it by little disks. This gives the following operad, which is one of the most important operads in this work.

Example 1.4.10. *Let $d \geq 1$ be an integer. The* little d-disks operad $B_d = \{B_d(k)\}_{k\geq 0}$ *was introduced by Boardman and Vogt in [4] as a tool for understanding d-fold loop spaces. It is the main example that historically motivated the introduction of operads [29]. For $k \geq 0$, the space $B_d(k)$ is defined to be the space of configurations of k closed d-disks with disjoints interiors contained in the unit disk D^d of the Euclidean space $\mathbb{R}^d$. The picture below is a typical element of $B_d(3)$. The insertion operation in B_d is defined in the obvious way.*



For more details about this operad, we refer the reader to [29, Chapter 4] or [8, Section 2]. Notice also that B_d is a suboperad of the coendomorphism operad $Coend_{D^d}$.

In next chapters we will see many other examples of operads such as the Kontsevich operad $\mathcal{K}_d$ (which will be well defined in Section 2.2.1), the Fulton-MacPherson operad $\mathcal{F}_d$ (Section 2.2.2), the cacti operad MS (Section 3.3.1), the operad $\mathcal{D}_2$ (Section 3.3.2), the operad $\widetilde{\mathcal{D}}_2$ (Section 3.4).
For a covariant monoidal functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ between symmetric monoidal categories, if $\mathcal{O}$ is an operad in $\mathcal{C}$, then it is easy to see that $F(\mathcal{O}) = \{F(\mathcal{O}(n))\}_{\geq 0}$ is an operad in $\mathcal{D}$. Hence the singular chains $S_*(\mathcal{K}_d)$, $S_*(\mathcal{F}_d)$, $S_*(B_d)$, the homologies $H_*(\mathcal{K}_d)$, $H_*(\mathcal{F}_d)$ are operads in chain complexes. For us the *commutative operad in chain complexes* , denoted by Com, is the operad $H_0(B_d)$, the 0th homology group of the little d-disks operad. This operad will appear in Chapter 5 in the study of the space of long links.

Example 1.4.11. *Let $\mathcal{A}s = \{\mathcal{A}s(n)\}_{n\geq 0}$ be the sequence defined by $\mathcal{A}s(n) = \mathbf{1}$ for each n, the unit for the tensor product of $\mathcal{C}$. It is easy to see that $\mathcal{A}s$ is a nonsymmetric operad, called the* associative operad .

Remark 1.4.12. *The object $\mathcal{A}s$ is cofibrant in the category $\mathcal{C}^{\mathbb{N}}$ because the unit object $\mathbf{1}$ is cofibrant in $\mathcal{C}$ by the unit axiom, which is a part of the definition (see Definition 1.3.3 or [13, Section 11.3.3]) of a symmetric monoidal model category.*

Certain operads has a distinguish operation in each arity. Such operads are called *multiplicative.* More precisely, we have the following definition, which will be extensively used in Chapters 2, 3, 4. Moreover, one of the main results (Theorem 4.1.1) of this thesis is expressed in terms of multiplicative operads.

Definition 1.4.13. *A* multiplicative operad *in $\mathcal{C}$ is a couple $(\mathcal{O}, \alpha)$ in which $\mathcal{O}$ is a nonsymmetric operad in $\mathcal{C}$ and $\alpha : \mathcal{A}s \longrightarrow \mathcal{O}$ is a morphism of nonsymmetric operads from the associative operad to $\mathcal{O}$.*

If there is no ambiguity, a multiplicative operad $(\mathcal{O}, \alpha)$ will be denoted just by $\mathcal{O}$. Notice that a multiplicative operad in toplogical spaces Top can be viewed as an operad in pointed topological spaces, since the associative operad in Top is $\mathcal{A}s = \{*\}_{n\geq 0}$. More generally, a multiplicative operad in $\mathcal{C}$ is an operad $\mathcal{O}$ with one special operation $\mu_n \in \mathcal{O}(n)$ in each arity (here we assume that objects in $\mathcal{C}$ are *living objects*) such that $\mu_p \circ_i \mu_q = \mu_{p+q-1}$ for all p, q and $1 \leq i \leq p$. The following remark says that it suffices to get one special operation in arity 0 and one other in arity 2 satisfying the same condition.

Remark 1.4.14. *A multiplicative structure on $\mathcal{O}$ is equivalent to having morphisms $e\colon \mathbf{1} \longrightarrow \mathcal{O}(0)$ and $\mu\colon \mathbf{1} \longrightarrow \mathcal{O}(2)$ satisfying*

$$\mu \circ_1 \mu = \mu \circ_2 \mu \quad \textit{and} \quad \mu \circ_1 e = \mu \circ_2 e = \mathbf{1}.$$

The following proposition is very easy.

Proposition 1.4.15. *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a symmetric monoidal functor. If $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ is a multiplicative operad in $\mathcal{C}$, then $F(\mathcal{O}) = \{F(\mathcal{O}(n))\}_{n\geq 0}$ is a multiplicative operad in $\mathcal{D}$.*

Here is some important construction concerning multiplicative operads. Note that this construction will appear in several places in this thesis.

Proposition 1.4.16. [31, Section 3] *To any multiplicative operad $\mathcal{O}$ in $\mathcal{C}$, one can associate a cosimplicial object $\mathcal{O}^\bullet$ in $\mathcal{C}$.*

Proof. Let $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ be a multiplicative operad in $\mathcal{C}$, and let $e\colon \mathbf{1} \longrightarrow \mathcal{O}(0)$ and $\mu\colon \mathbf{1} \longrightarrow \mathcal{O}(2)$ as in Remark 1.4.14. Define $\mathcal{O}^n = \mathcal{O}(n)$. Define also the cofaces morphisms $d^i\colon \mathcal{O}^n \longrightarrow \mathcal{O}^{n+1}$ and the codegeneracies morphims $s^j\colon \mathcal{O}^{n+1} \longrightarrow \mathcal{O}^n$ by the following formulas.

- For $0 \leq i \leq n+1$, d^i is the composite

$$d^i = \begin{cases} \mathcal{O}^n \xrightarrow{\cong} \mathbf{1} \otimes \mathcal{O}^n \xrightarrow{\mu\otimes id} \mathcal{O}^2 \otimes \mathcal{O}^n \xrightarrow{\circ_2} \mathcal{O}^{n+1} & \text{if } i = 0 \\ \mathcal{O}^n \xrightarrow{\cong} \mathcal{O}^n \otimes \mathbf{1} \xrightarrow{id\otimes\mu} \mathcal{O}^n \otimes \mathcal{O}^2 \xrightarrow{\circ_i} \mathcal{O}^{n+1} & \text{if } 1 \leq i \leq n \\ \mathcal{O}^n \xrightarrow{\cong} \mathbf{1} \otimes \mathcal{O}^n \xrightarrow{\mu\otimes id} \mathcal{O}^2 \otimes \mathcal{O}^n \xrightarrow{\circ_1} \mathcal{O}^{n+1} & \text{if } i = n+1. \end{cases}$$

- For $1 \leq j \leq n+1$, s^j is the composite

$$\mathcal{O}^{n+1} \xrightarrow{\cong} \mathcal{O}^{n+1} \otimes \mathbf{1} \xrightarrow{id\otimes e} \mathcal{O}^{n+1} \otimes \mathcal{O}^0 \xrightarrow{\circ_j} \mathcal{O}^n \ .$$

It is straightforward to check cosimplicial relations (1.2.8) with d^i and s^j thus defined. ☐

1.5 Homology spectral sequences

It is well known that a short exact sequence of chain complexes produces a long exact sequence in homology, which allows to compute the homology in a number of situations. On can generalize the notion of a short exact sequence by the one of a *filtered chain complex*. Associated to a chain complex with a filtration is called a *homology spectral sequence*, which is a gadget for computing the homology by "successive approximations", and which we define now.

Definition 1.5.1. *A* homology spectral sequence *or simply a* spectral sequence *consists of a sequence* $(E^r, d^r, \tau^r)_{r\geq r_0}$ *in which*

- r_0 *is a nonnegative integer;*

- $E^r = \{E^r_{p,q}\}_{p,q}$ *is a bigraded module called the* E^r page*;*

- $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ *is a differential of bidegree* $(-r, r-1)$*;*

- $\tau^r : H_*(E^r) \xrightarrow{\cong} E^{r+1}$ *is an isomorphism of bigraded modules.*

One can define in the similar way the notion of *cohomology spectral sequence* (the only difference is the fact that in cohomology, the differential d_r is of bidegree $(r, 1-r)$). For our purposes (as said the title of this section) we will consider only homology spectral sequences. The triple (E^r, d^r, τ^r) is sometimes just denoted by $\{E^r\}_{r\geq r_0}$ or more simply by $E^r_{p,q}$.

Definition 1.5.2. *Let* $\{E^r\}_{r\geq r_0}$ *be a spectral sequence. We say that it*

- converges *if for each* (p,q)*, there is an integer* $r_{p,q}$ *such that* $E^{r_{pq}}_{p,q} = E^{r_{pq}+i}_{p,q}$ *for all* $i \geq 0$*. In that case the* $E^\infty_{p,q}$ *term is defined by* $E^\infty_{p,q} = E^{r_{pq}}_{p,q}$*;*

- collapses at the E^r page *if* $d^k = 0$ *for all* $k \geq r$*. In that case the* E^∞ *page is defined by* $E^\infty = E^r$*.*

Let us explain now the construction that associates a spectral sequence to a chain complex with a filtration. A *filtered chain complex* is a triple $(A_*, d, \mathfrak{F})$ in which

- the pair (A_*, d) is a chain complex,

- $\mathfrak{F}$ is an increasing sequence

$$\mathfrak{F} = \cdots \subseteq F_{p-1}A_* \subseteq F_pA_* \subseteq F_{p+1}A_* \subseteq \cdots \subseteq A_*$$

  of subcomplexes of A_* such that $\cup_p F_pA_* = A_*$ and $\cap_p F_pA_* = \{0\}$ (such a $\mathfrak{F}$ is called a filtration),

- the differential preserves the filtration (that is, $d(F_pA_i) \subseteq F_pA_{i-1}$ for each i).

For a filtered chain complex $(A_*, d, \mathfrak{F})$, one associates a spectral sequence $\{E^r(A_*)\}_{r\geq 0}$ defined by

$$E^r_{p,q}(A_*) = \frac{\{z \in F_pA_{p+q} |\ dz \in F_{p-r}A_{p+q-1}\}}{F_{p-1}A_{p+q} + d(F_{p+r-1}A_{p+q+1})}. \tag{1.5.1}$$

The differential d induces a well-defined morphism $d^r \colon E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ such that $(d^r)^2 = 0$. For more nice explanations, see [30] or [12, Chapter 18]. Note that when the filtration is decreasing (in that case it is denoted by $\{F^pA_*\}_p$), we also have an induced spectral sequence defined as in (1.5.1).

Example 1.5.3. *Let $\mathbb{L}$ be the free Lie algebra generated by y, u, v, w all in degree 2, by z in degree 4 and by x in degree 7. Assume that $\mathbb{L}$ is equipped with the zero differential d (here d is of degree -1), and that $dx = [y, z] + [u, [v, w]]$. Consider now its suspension $s^{-1}\mathbb{L}$ (see formula (3.2.1) from Chapter 3 for the notion of suspension), and equip it with the zero product. It turns out to be a* Gerstenhaber algebra *(see Section 3.2.1 for the definition of a Gerstenhaber algebra) filtered by its* bracket-length*. We then have the associated spectral sequence, which collapses at*

$$E^2 = H_*(\mathbb{L}, d_1),\ \ d_1x = [y, z].$$

Since the differential d_1 is 0, it follows that the bracket $[y, z]$ vanishes at the E^2 page. Notice that it does not vanish in $s^{-1}\mathbb{L}$. Hence, the above spectral sequence does not collapse as a Gerstenhaber algebra.

The spectral sequence we look at in this work is the one of Bousfield-Kan in homology. To define it, we need to first recall the notion of the Dold-Kan normalization. Note that one of the main results (Theorem 5.1.6) in this thesis says that the homology Bousfield-Kan spectral sequence associated to a certain cosimplicial space collapses at the E^2 page.

1.5.1 The Dold-Kan normalization

Let $G_\bullet$ be a simplicial object in abelian groups (that is, a simplicial abelian group). Let

$$d_i \colon G_k \longrightarrow G_{k-1}, 0 \leq i \leq k, \text{ and } s_j \colon G_{k-1} \longrightarrow G_k, 1 \leq j \leq k$$

denote face and codegeneracy morphisms. Define a morphism $\partial \colon G_k \longrightarrow G_{k-1}$ by

$$\partial = \sum_{i=0}^{k} (-1)^i d_i.$$

It is straightforward, using simplicial relations from (1.2.3), to check that ∂ is actually a differential (that is, $\partial^2 = 0$). We thus get the following chain complex

$$G_* = G_0 \xleftarrow{\partial} G_1 \xleftarrow{\partial} G_2 \xleftarrow{\partial} \cdots.$$

This chain complex admits an interesting subcomplex N_*G defined by

$$N_*G = \begin{cases} G_0 & \text{if } \quad * = 0 \\ G_k \cap (\cap_{i=1}^k Ker\, d_i) & \text{if } \quad * = k \geq 1. \end{cases}$$

The chain complex N_*G is interesting because the inclusion $N_*G \hookrightarrow G_*$ induces an isomorphism in homology (see [16, Chapter III-Theorem 2.4]). Moreover, if we write $N(G_\bullet)$ for N_*G, the previous construction defines a functor

$$N\colon s\mathcal{A}b \longrightarrow \mathrm{Ch}_*$$

from simplicial abelian groups to chain complexes. This functor is called the *Dold-Kan normalization functor*, and it is an equivalence of categories (see [16, Chapter III-Corollary 2.3] or [56, Theorem 8.4.1]).

There exists another description (see [16, Chapter III-Theorem 2.1]) of N_*G given by

$$N_*G = \begin{cases} G_0 & \text{if } \quad * = 0 \\ G_k/(\sum_{j=1}^k im\, s_j) & \text{if } \quad * = k \geq 1. \end{cases}$$

We have the dual construction if one starts with a cosimplicial object $K^\bullet$ in abelian groups. If

$$d^i\colon K^{p-1} \longrightarrow K^p, 0 \leq i \leq p, \text{ and } s^j\colon K^p \longrightarrow K^{p-1}, 1 \leq j \leq p$$

denote the coface and codegeneracy morphisms, then the associated cochain complex is

$$(K^* = K^0 \xrightarrow{\delta} K^1 \xrightarrow{\delta} K^2 \xrightarrow{\delta} \cdots, \delta = \sum_{i=0}^p (-1)^i d^i).$$

The subcomplex $N^*K \subseteq K^*$ is defined by

$$K^* = \begin{cases} K^0 & \text{if } \quad * = 0 \\ K^p/(\sum_{i=1}^p im\, d^i) & \text{if } \quad * = p \geq 1. \end{cases}$$

It also admits an alternative description:

$$K^* = \begin{cases} K^0 & \text{if } \quad * = 0 \\ K^p \cap (\cap_{j=1}^p ker\, s^j) & \text{if } \quad * = p \geq 1. \end{cases}$$

Of course, the inclusion $N^*K \hookrightarrow K^*$ also induces an isomorphism in cohomology.

1.5.2 The homology Bousfield-Kan spectral sequence associated to a cosimplicial space

A good reference for this section is [6].

Let G be a cosimplicial object in the category of simplicial abelian groups. Then $(N^*N_*G, \partial, \delta)$ is a bicomplex called the *normalized bicomplex associated to G*. From that bicomplex, define a chain complex (TG, D) by

$$(TG)_n = \prod_{k\geq 0} N^k N_{k+n} G \qquad \text{and} \qquad D = \partial + (-1)^{n+1}\delta.$$

Here $D\colon (TG)n \longrightarrow (TG)_{n-1}$. Of course, it is very easy to check that $D^2 = 0$. The chain complex (TG, D) is called the *total complex associated to* G, and it admits a natural decreasing filtration

$$\cdots F^{m+1}TG \subseteq F^m TG \subseteq F^{m-1}TG \subseteq \cdots \subseteq F^0 TG$$

by the cosmplicial degree, which is explicitly defined by

$$(F^m TG)_n = \prod_{k\geq m} N^k N_{k+n} G.$$

One can also associate a total complex to any cosimplicial space $X^\bullet$. Let A be an abelian group. Recall that the tensor product $S\otimes A$ of a set S by A is the abelian group defined by $S\otimes A = \oplus_{s\in S} A$. So the tensor product $\mathrm{Sing}_\bullet(X^\bullet)\otimes A$ of $\mathrm{Sing}_\bullet(X^\bullet)$ by A cosimplicial degreewise is a cosimplicial simplicial abelian group. Here $\mathrm{Sing}_\bullet(-)\colon \mathrm{Top} \longrightarrow \mathrm{sSet}$ from toplogical spaces to simplicial sets is the functor defined in Example 1.2.2, and $\mathrm{Sing}_\bullet(X^\bullet) = \{\mathrm{Sing}_\bullet(X^n)\}_{n\geq 0}$ is a cosimplicial simplicial set. By applying now the previous construction with the cosimplicial simplicial abelian group $\mathrm{Sing}_\bullet(X^\bullet)\otimes A$, we obtain the filtered chain complex $T(\mathrm{Sing}_\bullet(X^\bullet)\otimes A)$.

Definition 1.5.4. *The spectral sequence associated to the filtered chain complex* $T(\mathrm{Sing}_\bullet(X^\bullet)\otimes A)$ *is called the* homology Bousfield-Kan spectral sequence *associated to* $X^\bullet$.

This spectral sequence will be denoted by $E^r\{S_*(X^\bullet; A)\}_{r\geq 0}$ or, if A is understood, by $E^r\{S_*(X^\bullet)\}_{r\geq 0}$. More generally, for a cosimplicial chain complex $C_*^\bullet$, the spectral sequence induced by the filtration by the cosimplicial degree will be denoted by $\{E^r(C_*^\bullet)\}_{r\geq 0}$. We will sometimes use the abreviation H_*BKSS for homology Bousfield-Kan spectral sequence. Under good assumptions [6], it converges to the homology $H_*(\mathrm{hoTot}X^\bullet)$ of the homotopy totalization of $X^\bullet$. The E^1 and E^2 pages of that spectral sequence have a nice description via the following isomorphisms from [6]

$$E^1_{p,q} = H_{q-p}(F^p/F^{p+1})T(A\otimes X) \cong N^p H_q(X; A) \simeq H_q(X^p),$$

and

$$E^2_{p,q} \cong H^p N^* H_q(X; A).$$

Here X denotes the cosimplicial simplicial set $\mathrm{Sing}_\bullet(X^\bullet)$. Notice that if we assume $p \leq 0$ and $q \geq 0$, then we have a second quadrant spectral sequence (the differential d^r is of course of bidegree $(-r, r-1)$) with $E^1_{p,q} \simeq H_q(X^{-p})$.

CHAPTER 2

Rational homology of $\overline{\mathbf{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$ as a vector space

2.1 Introduction

In this chapter, we essentially summarize some results obtained in the past several years in the study of the space of long knots. The two main results we look at are Theorem 2.3.3 (which says that Sinha's cosimplicial space $\mathcal{K}_d^\bullet$ is a cosimplicial model for the space of long knots) and Theorem 2.4.1 (which says that the H_*BKSS associated to $\mathcal{K}_d^\bullet$ collapses at the E^2 page rationally, when $d \geq 4$). As an immediate consequence of Theorem 2.4.1, there exists an isomorphism of vector spaces between the rational homology of the space of long knots and the E^2 page. In Chapter 4, we will prove that this isomorphism also respects the Gerstenhaber algebra structure (this is one of the main results of this thesis). A part from some proofs, nothing is new in this chapter.

Let us give now the definition of a *long knot*. Fix an integer $d \geq 1$, which represents the dimension of the ambient space, and let I be the interval defined by $I = [-1, 1]$. Let $a_1 = (0, \ldots, 0, 1)$ and $b_1 = (0, \cdots, 0, -1)$ be two fixed points in the opposite faces of the cube I^d, and let $\vec{S} = (0, \cdots, 0, -1) \in S^{d-1}$ denote the direction between a_1 and b_1. Fix a linear embedding $\epsilon \colon \mathbb{R} \hookrightarrow \mathbb{R}^d$ defined by $\epsilon(t) = (0, \cdots, 0, -t)$.

Definition 2.1.1. *A* long knot *is a smooth embedding $f \colon \mathbb{R} \hookrightarrow \mathbb{R}^d$ that satisfies the boundary conditions*

$$\begin{cases} f(I) \subseteq \mathbb{R}^{d-1} \times I & \\ f(t) = \epsilon(t) & \textit{if} \quad |t| \geq 1. \end{cases} \tag{2.1.1}$$

This definition implies that

- $f(-1) = a_1$, $f(1) = b_1$;
- $\frac{f'(-1)}{\|f'(-1)\|} = \frac{f'(1)}{\|f'(1)\|} = \vec{S}$.

One can define a long knot as a smooth embedding $f\colon \mathbb{R} \hookrightarrow \mathbb{R}^d$ that coincides outside any fixed compact set with a fixed linear embedding. It is clear that this definition is equivalent to Definition 2.1.1. Notice that the case $d = 1$ or $d = 2$ is not so interesting because every long knot is then isotopic to the trivial long knot. This is the reason for which we will assume $d \geq 3$, unless mentioned otherwise.

One of the main objects of our study is the *space of long knots*. Before defining it, we recall the notion of *weak $\mathcal{C}^\infty$-topology* (a good reference for that topology is Hirsch [21, pages 34-35]). Let M and N be two $\mathcal{C}^k$ manifolds, $0 \leq k \leq \infty$, and let $\mathcal{C}^k(M, N)$ denote the collection of $\mathcal{C}^k$ maps from M to N. For two charts $(U, \phi) \subseteq M$ and $(V, \psi) \subseteq N$, for a compact subset $K \subseteq U$, for $f \in \mathcal{C}^k(M, N)$ such that $f(K) \subseteq V$, for $0 < \delta \leq \infty$, define a weak subbasic neighborhood

$$\mathcal{N} := \mathcal{N}^k(f; (U, \phi), (V, \psi), K, \delta)$$

to be the set of $\mathcal{C}^k$ maps $g\colon M \longrightarrow N$ such that $g(K) \subseteq V$ and

$$\left\| D^p(\psi f \phi^{-1})(x) - D^p(\psi g \phi^{-1})(x) \right\| < \delta$$

for all $x \in \phi(K)$, and for all $p \in \{0, \cdots, k\}$.

(WT) The *weak $\mathcal{C}^k$-topology* on the collection $\mathcal{C}^k(M, N)$ is generated by sets $\mathcal{N}$. On $\mathcal{C}^\infty(M, N)$, it is called the *weak $\mathcal{C}^\infty$-topology*.

Definition 2.1.2. *The* space of long knots *is the collection*

$$\{f\colon \mathbb{R} \hookrightarrow \mathbb{R}^d \textit{ such that } f \textit{ is a long knot}\}$$

*endowed with the weak $\mathcal{C}^\infty$-topology described by **(WT)**.*

The space of long knots will be denoted by $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$. We put the letter "c" in subscript to make the difference with the classical space $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$ of all smooth embeddings of $\mathbb{R}$ in $\mathbb{R}^d$. One can read $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$ as the space of compactly supported smooth embeddings of $\mathbb{R}$ inside $\mathbb{R}^d$. One of our goals, as mentioned in the introduction of this thesis, is to study this latter space. More precisely, we want to understand its rational homology.

In [42], Shubert proves that the connected-sum of two long knots induces a commutative monoidal structure on the path-components, $\pi_0\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$, of $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$. His idea for the commutativity is the fact that one can "pull one knot through another", and this suggests the existence of a map $S^1 \times (\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d))^2 \longrightarrow \mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$. It is known that such a map would exist if the connected-sum operation was induced by the action of the little 2-disks operad on $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$. Unfortunately, this latter operad acts only up to homotopy (in the sense that certain diagrams are only commutative up to homotopy), and this poses a problem. To get around it, one works with the space of "thickened long knots" or "long knots modulo immersions", which we define now.

Definition 2.1.3. • *A* long immersion *is a smooth immersion $f\colon \mathbb{R} \hookrightarrow \mathbb{R}^d$ satisfying boundary conditions (2.1.1).*

- *The* space of long immersions *is the collection*

$$\{f\colon \mathbb{R} \hookrightarrow \mathbb{R}^d \textit{ such that } f \textit{ is a long immersion}\}$$

endowed with the weak $\mathcal{C}^\infty$*-topology described by* ***(WT)*** *above. We denote it by* $\mathrm{Imm}_c(\mathbb{R}, \mathbb{R}^d)$.

- *The* space of long knots modulo immersions *is defined to be the homotopy* fiber *of the inclusion* $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d) \hookrightarrow \mathrm{Imm}_c(\mathbb{R}, \mathbb{R}^d)$*, and it is denoted by* $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$.

Remark 2.1.4. *By abuse of terminology, in the rest of this thesis, we will sometimes say "long knot" in the place of "long knot modulo immersions".*

Now we have two spaces: $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$ and $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$. The natural question one can ask is to know if they are at least weakly equivalent. Although this question has a negative answer, they are at least related by the weak equivalence (2.1.2), which is obtained as follows. It is known that by Smale-Hirsch theory [46], there is a weak equivalence $\mathrm{Imm}_c(\mathbb{R}, \mathbb{R}^d) \simeq \Omega S^{d-1}$. It is also known that the inclusion $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d) \hookrightarrow \mathrm{Imm}_c(\mathbb{R}, \mathbb{R}^d)$ is null-homotopic [44, Proposition 5.17]. Therefore, there is a weak equivalence

$$\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d) \simeq \mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d) \times \Omega^2 S^{d-1}. \tag{2.1.2}$$

From (2.1.2), to understand the rational homology of $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$, it suffices to understand the one of $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$. This is because the rational homology of $\Omega^2 S^{d-1}$ is very simple (it is isomorphic to a free graded commutative algebra on one or two generators depending of the parity of d).

The main result of [44] due to Sinha states that the space $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$ is weakly equivalent to the homotopy totalization $\mathrm{hoTot}\mathcal{K}_d^\bullet$ of the Sinha cosimplicial space (see Theorem 2.3.3). Therefore, one can understand the rational homology of $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$ by the mean of the H_*BKSS associated to $\mathcal{K}_d^\bullet$.

Outline of the chapter

- In Section 2.2 we review the Kontsevich operad $\mathcal{K}_d$, the Fulton and MacPherson operad $\mathcal{F}_d$, and the operad of admissible diagrams $\mathcal{D}_d^\vee$. We also show that $\mathcal{K}_d$ and $\mathcal{D}_d^\vee$ are multiplicative operads while $\mathcal{F}_d$ is only an "up to homotopy" multiplicative operad. All these results will be used in the proof of Theorem 4.1.1 (from Chapter 4), which is one of the most important results in this thesis.

- In Section 2.3 we first define the Sinha cosimplcial space $\mathcal{K}_d^\bullet$. Then we review the proof of Theorem 2.3.3, which says that $\mathcal{K}_d^\bullet$ is actually a cosimplicial model for the space of long knots.

- In Section 2.4 we state the main result of [26, Theorem 1.2] due to Lambrechts, Turchin and Volić. Next we recall the definition of formality for morphisms of operads. Finally we state the relative formality theorem of the little d-disks operad B_d, which will be used in Section 4.3 from Chapter 4.

2.2 The Kontsevich operad $\mathcal{K}_d$, the Fulton-MacPherson operad $\mathcal{F}_d$ and the operad of admissible diagrams $\mathcal{D}_d^\vee$

In this section we review operads $\mathcal{K}_d$, $\mathcal{F}_d$ and $\mathcal{D}_d^\vee$.

2.2.1 The Kontsevich operad $\mathcal{K}_d$

Here we first define $\mathcal{K}_d$, for $d \geq 1$. Then we make the observation that $\mathcal{K}_1$ is the associative operad in topological spaces. Finally we show that $\mathcal{K}_d$ is a multiplicative operad.

In order to define the space $\mathcal{K}_d(k)$ we need to introduce the *configuration space* of k points in $\mathbb{R}^d$. For two sets E and F, we denote by $\mathrm{Inj}(E, F)$ the set of injective maps from E to F. For $k \geq 0$ be an integer, define a natural ordered set $\underline{k} = \{1, \cdots, k\}$.

Definition 2.2.1. *The* configuration space *of k points in $\mathbb{R}^d$, denoted by* $\mathrm{Conf}(k, \mathbb{R}^d)$*, is the space defined by*

$$\mathrm{Conf}(k, \mathbb{R}^d) = \mathrm{Inj}(\underline{k}, \mathbb{R}^d).$$

It is topologized as a subspace of the product $(\mathbb{R}^d)^k = \prod_{i=1}^k \mathbb{R}^d$.

For $k \geq 1$ and for $x \colon \underline{k} \hookrightarrow \mathbb{R}^d$ be a configuration of k points in $\mathbb{R}^d$, we set $x = (x_1, \cdots, x_k)$, where $x_i = x(i)$. So an element of $\mathrm{Conf}(k, \mathbb{R}^d)$ can be written on the form $(x_1, \cdots, x_k)$.

We also need (again in order to define $\mathcal{K}_d(k)$) to define a map. Define first, for $i, j \in \underline{k}$ such that $1 \leq i < j \leq k$, a map $\alpha_{ij} \colon \mathrm{Conf}(k, \mathbb{R}^d) \longrightarrow S^{d-1}$ by

$$\alpha_{ij}(x_1, \cdots, x_k) = \frac{x_j - x_i}{\|x_j - x_i\|}.$$

Notice that $\alpha_{ij}(x_1, \cdots, x_k)$ is nothing other than the direction between the points x_i and x_j with $i < j$. Next define a map α by

$$\alpha = (\alpha_{ij})_{1 \leq i < j \leq k} \colon \mathrm{Conf}(k, \mathbb{R}^d) \longrightarrow \prod_{1 \leq i < j \leq k} S^{d-1}. \tag{2.2.1}$$

Definition 2.2.2. *The space $\mathcal{K}_d(k)$ ($d \geq 1$) is defined to be the closure of the image of α, and is called the* Kontsevich compactification *of the configuration space* $\mathrm{Conf}(k, \mathbb{R}^d)$*,*

$$\mathcal{K}_d(k) = \overline{\mathrm{im}(\alpha)}.$$

Remark 2.2.3. *If $x = (x_1, x_2, x_3)$ is a configuration of three aligned points in $\mathbb{R}^d$, the components of α are all the same on x (that is, $\alpha_{12}(x) = \alpha_{13}(x) = \alpha_{23}(x)$). This implies that the map α is not injective, but Sinha [43, Theorem 5.10] shows that there exists a weak equivalence*

$$\mathcal{K}_d(k) \simeq \mathrm{Conf}(k, \mathbb{R}^d). \tag{2.2.2}$$

Remark 2.2.4. *The space $\mathcal{K}_d(0)$, as well as the space $\mathcal{K}_d(1)$, is the one point space (this comes immediately from Definition 2.2.2).*

Intuitively, one should think an element $x \in \mathcal{K}_d(k)$ as a "virtual" configuration of k points in which it is possible to get two or more points that are infinitesimally close to each other in such a way that the direction between any two of them is always well recorded. From this point of view, given two virtual configurations $x = (x_1, \cdots, x_p) \in \mathcal{K}_d(p)$ and $y = (y_1, \cdots, y_q) \in \mathcal{K}_d(q)$, given an integer $1 \leq i \leq p$, we can form a new virtual configuration $x \circ_i y \in \mathcal{K}_d(p+q-1)$ of $p+q-1$ points by "substituting" the point x_i by the configuration y made infinitesimal, and by keeping the other points of the configuration x. This defines an operad structure. To be more precise, let $\mathcal{B}_d(k)$ be the space defined by

$$\mathcal{B}_d(k) = (S^{d-1})^{\frac{k(k-1)}{2}}.$$

By convention we set $\mathcal{B}_d(0) = *$ and $\mathcal{B}_d(1) = *$. Then the collection $\{\mathcal{B}_d(k)\}_{k\geq 0}$ is a topological operad [44]. Its structure

$$\circ_r \colon \mathcal{B}_d(p) \times \mathcal{B}_d(q) \longrightarrow \mathcal{B}_d(p+q-1), 1 \leq r \leq p,$$

is defined by

$$(\alpha \circ_r \beta)_{ij} = \begin{cases} \alpha_{ij} & \text{if} \quad i < j \leq r \\ \beta_{i-r+1, j-r+1} & \text{if} \quad r \leq i < j \leq r+q-1 \\ \alpha_{i-q+1, j-q+1} & \text{if} \quad r+q \leq i < j \\ \alpha_{ir} & \text{if} \quad i < r \leq j < r+q \\ \alpha_{rj} & \text{if} \quad r \leq i < r+q \leq j. \end{cases} \tag{2.2.3}$$

Theorem 2.2.5. [44, Definition 4.1 and Theorem 4.5] *For $d \geq 1$, the collection $\mathcal{K}_d(\bullet) = \{\mathcal{K}_d(k)\}_{k\geq 0}$ forms a suboperad of $\mathcal{B}_d$.*

Definition 2.2.6. *The operad $\mathcal{K}_d(\bullet)$, $d \geq 1$, is called the* Kontsevich operad *. We will denote it just by $\mathcal{K}_d$.*

Remark 2.2.7. *Since there is only one direction on the real line $\mathbb{R}$, it follows that $\mathcal{K}_1$ is the associative operad in topological spaces. That is,*

$$\mathcal{K}_1 = \mathcal{A}s. \tag{2.2.4}$$

Now we are going to make an important observation (the following proposition) about the Kontsevich operad. At the beginning of the introduction of this chapter, we have defined two points: a_1 and b_1. Consider these two points and the direction $\vec{S} = (0, \cdots, 0, -1)$ between them. By definition, the map α from (2.2.1) sends the configuration $(a_1, b_1) \in \mathrm{Conf}(2, \mathbb{R}^d)$ to $\vec{S}$. Therefore, the unit vector $\vec{S}$ is an element of $\mathcal{K}_d(2)$ by the definition of $\mathcal{K}_d(2)$. Define now $\mu \in \mathcal{K}_d(2)$ by $\mu = \vec{S}$, and define also $e \in \mathcal{K}_d(0)$ to be the unique element of $\mathcal{K}_d(0) = *$. Then, recalling the definition of a multiplicative operad from Definition 1.4.13, we have the following proposition.

Proposition 2.2.8. [44] *For $d \geq 1$ the Kontsevich operad $\mathcal{K}_d$ is a multiplicative operad.*

Proof. By (2.2.3) the three components of $\mu \circ_1 \mu \in (S^{d-1})^3$ are the same (they are all equal to $\vec{S}$). Similarly, the three components of $\mu \circ_2 \mu$ are all equal to $\vec{S}$. Therefore we have the equality

$$\mu \circ_1 \mu = \mu \circ_2 \mu. \tag{2.2.5}$$

On the other hand, if id denotes the unique element of $\mathcal{K}_d(1) = *$, we have the following equality

$$\mu \circ_1 e = \mu \circ_2 e = \mathrm{id}. \tag{2.2.6}$$

Using now equations (2.2.5), (2.2.6) and Remark 1.4.14 from Chapter 1, the desired result follows. □

We end this section with the following theorem (due to Sinha), which connects operads B_d and $\mathcal{K}_d$.

Theorem 2.2.9. [43] *The Kontsevich operad $\mathcal{K}_d$ and the little d-disks operad B_d are weakly equivalent as topological operads. That is,*

$$\mathcal{K}_d \simeq B_d.$$

2.2.2 The Fulton-MacPherson operad $\mathcal{F}_d$

Here we first define the operad $\mathcal{F}_d$ for $d \geq 1$. Next we show that it is not a multiplicative operad, but an "up to homotopy multiplicative operad".

The Fulton-MacPherson operad [15, 39] was simultaneously introduced by several people, in particular by Getzler and Jones in [15]. However we suggest [27, Chapter 5] for more details about $\mathcal{F}_d$. It is defined in the "similar way" as the Kontsevich operad $\mathcal{K}_d$ (see Section 2.2.1) except that one also takes in consideration the relative distance between three points taken in a configuration of points. So an element $x \in \mathcal{F}_d(p)$ can be viewed as a "‘virtual" configuration of p points in which points are allowed to be infinitesimally close to each other while the directions and the relative distances of their approach are recorded. To be more precise, consider the map

$$\beta \colon \mathrm{Conf}(p, \mathbb{R}^d) \longrightarrow \left(\prod_{1 \leq i < j \leq p} S^{d-1} \right) \times \left(\prod_{1 \leq i < j < k \leq p} [0, +\infty] \right)$$

defined by

$$\beta(x) = (\beta_{ij}(x), \delta_{ijk}(x)),$$

where (for $x = (x_1, \cdots, x_p) \in \mathrm{Conf}(p, \mathbb{R}^d)$)

$$\beta_{ij}(x) = \frac{x_j - x_i}{\|x_j - x_i\|} \qquad \text{and} \qquad \delta_{ijk}(x) = \frac{\|x_i - x_j\|}{\|x_i - x_k\|}.$$

Notice that the vector $\beta_{ij}(x)$ thus defined gives the direction between the points x_i and x_j, whereas the real number $\delta_{ijk}(x)$ gives the relative distance between the points x_i, x_j and x_k.

Definition 2.2.10. *The space $\mathcal{F}_d(p)$ is defined to be the closure of the image of β, and is called the* Fulton-MacPherson compactification *of the configuration space* $\mathrm{Conf}(p, \mathbb{R}^d)$*,*

$$\mathcal{F}_d(p) = \overline{\mathrm{im}(\beta)}.$$

The following proposition is proved in the similar way as Theorem 2.2.5.

Proposition 2.2.11. [44, Section 4] *For $d \geq 1$, the collection of spaces $\mathcal{F}_d(\bullet) = \{\mathcal{F}_d(p)\}_{p\geq 0}$ is an operad. We will denote it just by $\mathcal{F}_d$.*

Definition 2.2.12. *The operad $\mathcal{F}_d$ is called the* Fulton-MacPherson operad*.*

The operad structure of $\mathcal{F}_d$ (the reader can refer to [27, Chapter 5] for more details about this structure) is like that of the Kontsevich operad $\mathcal{K}_d$, which was defined in Section 2.2.1. In that section, we have seen that the Kontsevich operad is multiplicative. Since these two operads are defined in the "similar way", one could ask whether the Fulton-MacPherson operad is also multiplicative. The following proposition gives a negative answer.

Proposition 2.2.13. *For $d \geq 1$, the Fulton-MacPherson operad $\mathcal{F}_d$ is not a multiplicative operad, but it is an up to homotopy multiplicative operad in topological spaces.*

For the meaning of "up to homotopy multiplicative operad" , see Definition 4.2.1 from Chapter 4.

Proof. For $d \geq 1$ there is a natural morphism $\varphi_d \colon \mathcal{F}_d \longrightarrow \mathcal{K}_d$ in the category of topological operads, which is defined by the projection on the first component. It is straightforward to see that this map turns out to be a weak equivalence. In particular, we have the weak equivalence $\varphi_1 \colon \mathcal{F}_1 \xrightarrow{\sim} \mathcal{K}_1 = \mathcal{A}s$.
On the other hand, the inclusion $\mathbb{R} \hookrightarrow \mathbb{R}^d$ induces two morphisms of operads:

$$\mu \colon \mathcal{K}_1 \longrightarrow \mathcal{K}_d \qquad \text{and} \qquad \eta \colon \mathcal{F}_1 \longrightarrow \mathcal{F}_d.$$

We thus form the commutative square

$$\begin{tikzcd} \mathcal{K}_1 \arrow[r, "\mu"] & \mathcal{K}_d \\ \mathcal{F}_1 \arrow[u, "\sim", "\varphi_1"'] \arrow[r, "\eta"] & \mathcal{F}_d. \arrow[u, "\sim", "\varphi_d"'] \end{tikzcd} \tag{2.2.7}$$

This square tells us that the Fulon-MacPherson operad is not a multiplicative operad, but an up to homotopy multiplicative operad. Of course, by definition, the operad $\mathcal{F}_1$ is not homeomorphic to $\mathcal{K}_1$. For instance the space $\mathcal{F}_1(3)$ is homeomorphic to the interval I, whereas the space $\mathcal{K}_1(3)$ is the one point space. $\square$

2.2.3 The operad of admissible diagrams $\mathcal{D}_d^\vee$

The goal of this section is to recall the construction (for $d \geq 2$) of the operad $\mathcal{D}_d^\vee(\bullet) = \{\mathcal{D}_d^\vee(k)\}_{k\geq 0}$, which is the dual of the cooperad $\mathcal{D}_d(\bullet)$ of admissible diagrams. The case $d = 1$ is trivial and will be mentioned at the end of the section. We show that $\mathcal{D}_1^\vee$ is the associative operad, and $\mathcal{D}_d^\vee$ is a multiplicative operad in chain complexes. This section is based on [27, Chapter 6 and Chapter 7].

Definition 2.2.14. *A* diagram $\mathcal{G}$ *consists of the following data:*

- *a finite set* $A_{\mathcal{G}}$ *whose elements are called* external vertices*;*
- *an ordered finite set* $I_{\mathcal{G}}$ *disjoint from* $A_{\mathcal{G}}$*, and whose elements are called* internal vertices*;*
- *an ordered finite set* $E_{\mathcal{G}}$ *whose elements are called* edges*;*
- *two functions* $s_{\mathcal{G}}, t_{\mathcal{G}} \colon E_{\mathcal{G}} \longrightarrow \colon A_{\mathcal{G}} \coprod I_{\mathcal{G}}$ *called the* source function *and the* target function *respectively.*

The following picture is an example of a diagram. The set of external vertices is $\{1, 2, 3, 4, 5\}$, and the set of internal vertices is $\{6, \cdots, 15\}$.

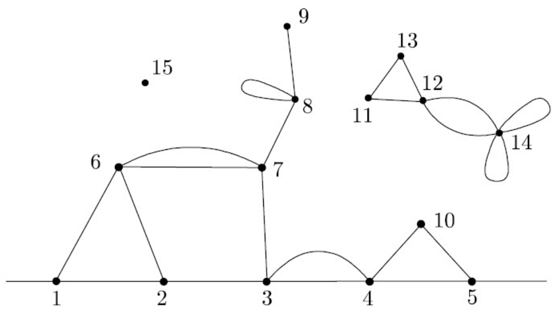


Figure 2.1: An example of a diagram

We will say that two diagrams $\mathcal{G}$ and $\mathcal{G}'$ are *isomorphic* if $A_{\mathcal{G}} = A_{\mathcal{G}'}$ and there exists two order-preserving bijections $\varphi_E \colon E_{\mathcal{G}} \xrightarrow{\cong} E_{\mathcal{G}'}$ and $\varphi_I \colon I_{\mathcal{G}} \xrightarrow{\cong} I_{\mathcal{G}'}$ such that $\varphi_V \circ s_{\mathcal{G}} = s_{\mathcal{G}'} \circ \varphi_E$ and $\varphi_V \circ t_{\mathcal{G}} = t_{\mathcal{G}'} \circ \varphi_E$, where $\varphi_V := id_{A_{\mathcal{G}}} \coprod \varphi_I \colon V_{\mathcal{G}} \xrightarrow{\cong} V_{\mathcal{G}'}$ is the obvious extension of φ_I. Notice that the finite set $A_{\mathcal{G}}$ is sometimes taken to be the set $\underline{k} = \{1, \cdots, k\}$. In that case $\mathcal{G}$ is called *diagram with k external vertices.*

Definition 2.2.15. *Let* $\mathbb{K}$ *be a field of characteristic* 0*. For* $d \geq 2$ *and* $k \geq 0$*, define* $\widehat{\mathcal{D}}_d(k)$ *to be the free* $\mathbb{K}$*-module generated by isomorphism classes of diagrams with k external vertices modulo some signed relations when the order of internal vertices or edges is permuted, or the orientation of some edge is reversed (for a precise definition see [27, Definition 6.2.2]).*

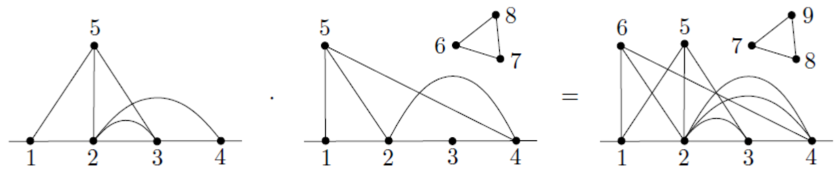
We put on the module $\widehat{\mathcal{D}}_d(k)$ a grading defined by the formula

$$\deg(\mathcal{G}) = |E_\mathcal{G}|\,(d-1) - |I_\mathcal{G}|\,d, \tag{2.2.8}$$

where $|E_\mathcal{G}|$ is the number of edges and $|I_\mathcal{G}|$ the number of internal vertices. The module $\widehat{\mathcal{D}}_d(k)$ is also equipped with a product and a differential, which we define now. Let us begin with the product. Let $\mathcal{G}'$ and $\mathcal{G}''$ be two diagrams with the same set A of external vertices. We can assume, without lost the generality, that the sets $I'_\mathcal{G}$ and $I''_\mathcal{G}$ are disjoint as well as the sets $E'_\mathcal{G}$ and $E''_\mathcal{G}$. The product $\mathcal{G} = \mathcal{G}'\mathcal{G}''$ is the diagram with

- A as the set of external vertices;
- $I_\mathcal{G} := I'_\mathcal{G} \coprod I''_\mathcal{G}$ as the set of internal vertices;
- $E_\mathcal{G} := E'_\mathcal{G} \coprod E''_\mathcal{G}$ as the set of edges;
- $s_\mathcal{G} = s_{\mathcal{G}'} \coprod s_{\mathcal{G}''}$ as the source function and $t_\mathcal{G} = t_{\mathcal{G}'} \coprod t_{\mathcal{G}''}$ as the target function.

Here is an example of a product of two diagrams.



Let $\mathbf{1}$ denote the diagram with $\underline{k}$ as the set external vertices, the empty set as the set of internal vertices and the set of edges. It is easy to check that this special diagram is the unit for the previous product. It is then call the *unit diagram.*

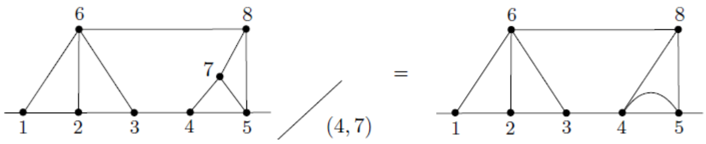
Proposition 2.2.16. [27, Proposition 6.3.2] *The module $\widehat{\mathcal{D}}_d(k)$, equipped with the above product, turns out to be a commutative $\mathbb{Z}$-graded algebra, when $d \geq 2$.*

Define now the differential. Let $\mathcal{G}$ be a diagram, for example the diagram in Figure 2.1. Before defining the differential of $\mathcal{G}$, we recall four notions:

- a *loop* in $\mathcal{G}$ is an edge whose endpoints are identical. For instance, in Figure 2.1, we have three loops, at vertices 14 and 8;
- a *chord* in $\mathcal{G}$ is an edge connecting two distinct external vertices. In Figure 2.1, there is only one chord, $(3, 4)$;
- a *dead end* is an edge that is not a loop and such that at least one if its endpoints is internal and has only one adjacent vertex. In our example, we have three dead ends, $(8, 9), (12, 14)_1$, and $(12, 14)_2$;

- a *contractible edge* is an edge that is neither a chord, nor a loop, nor a dead end. For example, again in Figure 2.1, we have two double contractible edges $(6,7)_1$ and $(6,7)_2$; and nine other simple contractible edges.

Roughly speaking, the differential of $\mathcal{G}$ is an alternating sum of diagrams obtained from $\mathcal{G}$ by contracting the contractible edges (a precise definition is given in [27, Formula (6.1)]). The following picture is a contraction of an edge.



By endowing the module $\widehat{\mathcal{D}}_d(k)$ with this differential, we get the following theorem.

Theorem 2.2.17. [27, Theorem 6.4.7] *For $k \geq 0$ and $d \geq 2$, the module $\widehat{\mathcal{D}}_d(k)$ is a commutative differential $\mathbb{Z}$-graded algebra.*

Let us define now the notion of *admissible diagram.* Recall that *double edges* are distinct edges that have the same set of endpoints.

Definition 2.2.18. [27, Definition 6.5.1]

- *An* admissible diagram *is a diagram that contains no double edges, no loops, no internal vertices of valence less than or equal to two, and if each of its internal vertices is connected to some external vertex.*

- *A* non-admissible diagram *is a diagram, which is not admissible.*

The Figure 2.2 below is an example of an admissible diagram.

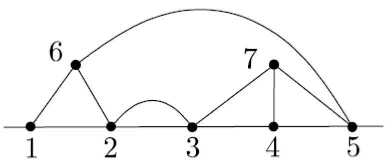


Figure 2.2: An admissible diagram

Definition 2.2.19. *For $k \geq 0$ and $d \geq 2$, the commutative differential $\mathbb{Z}$-graded algebra $\mathcal{D}_d(k)$ of admissible diagrams is defined to be the quotient of $\widehat{\mathcal{D}}_d(k)$ by the ideal generated by the non-admissible diagrams.*

Theorem 2.2.20. [27, Theorem 7.4.3] *The collection $\mathcal{D}_d(\bullet) = \{\mathcal{D}_d(k)\}_{k\geq 0}$ is a cooperad in commutative differential graded algebras ($\mathbb{Z}$ graded if $d = 2$).*

Proof. We will just give an idea of the proof. In particular we will explain the cooperad structure with an explicit example. For the full proof we refer the reader to [27, Chapter 7].

Let $\nu\colon \underline{k} \longrightarrow P$ be a map (called *weak ordered partition* in [27]) from $\underline{k} = \{1, \cdots, k\}$ to an ordered set P. Take for instance $\nu\colon \underline{5} \longrightarrow \{\alpha, \beta\}$, with $\alpha < \beta$, defined by

$$\nu(v) = \begin{cases} \alpha & \text{if} \quad v = 1,2,3 \\ \beta & \text{if} \quad v = 4,5. \end{cases}$$

We want to build cooperad structure maps

$$\Psi_\nu \colon \mathcal{D}_d(k) \longrightarrow \mathcal{D}_d(P) \otimes \bigotimes_{p \in P} \mathcal{D}_d(k_p),$$

where $k_p = \nu^{-1}(p)$. The plan is to build maps

$$\widehat{\Psi}_\nu \colon \widehat{\mathcal{D}}_d(k) \longrightarrow \widehat{\mathcal{D}}_d(P) \otimes \bigotimes_{p \in P} \widehat{\mathcal{D}}_d(k_p)$$

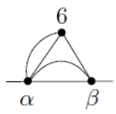
inducing the linear maps Ψ_ν. We will do it in our example. That is, we will build a map

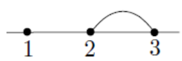
$$\widehat{\Psi}_\nu \colon \widehat{\mathcal{D}}_d(5) \longrightarrow \widehat{\mathcal{D}}_d(2) \otimes (\widehat{\mathcal{D}}_d(3) \otimes \widehat{\mathcal{D}}_d(2)).$$

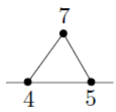
Let $\mathcal{G}$ be the diagram of Figure 2.2 above. Let $\lambda\colon \underline{7} = V_{\mathcal{G}} \longrightarrow P^* = \{0, \alpha, \beta\}$ be a map (called *condensation of* $V_{\mathcal{G}}$ *relative to* ν, that is, $\lambda | A_{\mathcal{G}} = \nu$) defined by

$$\lambda(v) = \begin{cases} \alpha & \text{if} \quad v = 1,2,3 \\ \beta & \text{if} \quad v = 4,5,7 \\ 0 & \text{if} \quad v = 6. \end{cases}$$

Define $\mathcal{G}(\lambda) = \pm \mathcal{G}(\lambda, 0) \otimes \mathcal{G}(\lambda, \alpha) \otimes \mathcal{G}(\lambda, \beta)$ by the following pictures.

$\mathcal{G}(\lambda, 0) =$ 

$\mathcal{G}(\lambda, \alpha) =$ 

$\mathcal{G}(\lambda, \beta) =$ 

Then $\widehat{\Psi}_\nu(\mathcal{G})$ is defined by

$$\widehat{\Psi}_\nu(\mathcal{G}) = \sum_\lambda \mathcal{G}(\lambda),$$

where λ runs over condensations of $V_{\mathcal{G}}$ relative to ν.

$\square$

We are now ready to define $\mathcal{D}_d^\vee$.

Definition 2.2.21. *The* operad of admissible diagrams , *denoted by* $\mathcal{D}_d^\vee(\bullet)$ *or simply by* $\mathcal{D}_d^\vee$*, is defined to be the dual of the cooperad* $\mathcal{D}_d(\bullet)$*.*

Let us treat now the trivial case $d = 1$. In that case, we set for $k \geq 0$

$$\mathcal{D}_1(k) = H^*(\mathcal{F}_1(k)), \tag{2.2.9}$$

and define $\mathcal{D}_1^\vee(k)$ to be the dual of $\mathcal{D}_1(k)$. It is clear that $\mathcal{D}_1^\vee$ is the associative operad in chain complexes. That is,

$$\mathcal{D}_1^\vee = \mathcal{A}s. \tag{2.2.10}$$

Proposition 2.2.22. *For $d \geq 1$ the operad $\mathcal{D}_d^\vee$ is a multiplicative operad in chain complexes.*

Proof. For $k \geq 0$, define the morphism $\psi_k \colon \mathcal{D}_d(k) \longrightarrow \mathcal{D}_1(k)$ by

$$\psi_k(\mathcal{G}) = \begin{cases} \mathbf{1} & \text{if } \mathcal{G} = \mathbf{1} \text{ the unit diagram} \\ 0 & \text{otherwise.} \end{cases}$$

By [27, Lemma 10.1], the morphism $\psi = \{\psi_k\}_{k\geq 0} \colon \mathcal{D}_d \longrightarrow \mathcal{D}_1$ is a morphism of cooperads in CDGA. Therefore the dual morphism $\psi^\vee \colon \mathcal{D}_1^\vee \longrightarrow \mathcal{D}_d^\vee$ is a morphism of operads in chain complexes. Since $\mathcal{D}_1^\vee = \mathcal{A}s$ (by (2.2.10) above), the desired result follows. □

2.3 Sinha's cosimplicial model for the space of long knots

In this section we first define the Sinha cosimplicial space $\mathcal{K}_d^\bullet$. Next we review the proof of the fact that it gives a cosimplicial model for the space of long knots (see [44] for full details about this proof).

In [32, Section 10], McClure and Smith show that to any multiplicative operad $\mathcal{O}$ in toplogical spaces is associated a cosimplicial space $\mathcal{O}^\bullet$ (recall that this construction has been done in Proposition 1.4.16 from Chapter 1 for multiplicative operads in any symmetric monoidal category). In the particular case of the Kontsevich operad $\mathcal{K}_d$, which is a multiplicative operad (by Proposition 2.2.8) in topological spaces, this construction gives rise to a cosimplicial space denoted by $\mathcal{K}_d^\bullet$.

Definition 2.3.1. *The* Sinha cosimplicial space *is the cosimplicial space $\mathcal{K}_d^\bullet$.*

Remark 2.3.2. *Let $\vec{v}$ be a fixed unit vector in $\mathbb{R}^d$. The coface map $d^i \colon \mathcal{K}_d^k \longrightarrow \mathcal{K}_d^{k+1}$ consists of doubling the ith point (the direction between the ith point and the new one is taken to be $\vec{v}$), when $1 \leq i \leq k$. However, the coface maps d^0 and d^{k+1} consist of adding a point at the infinity (at $+\infty$ for d^0 and at $-\infty$ for d^{k+1}). If all the points of the "virtual" configuration $x \in \mathcal{K}_d^k$ are inside the cube I^d, then the coface d^0 consists of inserting a_0 whereas d^{k+1} consists of inserting a_1. The codegeneracy map $s^j \colon \mathcal{K}_d^{k+1} \longrightarrow \mathcal{K}_d^k$ consists of forgetting the jth point in the configuration and relabeling appropriately.*

The following theorem, due to Sinha, is the main result of [44].

Theorem 2.3.3. [44] *For $d \geq 4$, the Sinha cosimplicial space $\mathcal{K}_d^\bullet$ gives a cosimplicial model for the space of long knots . That is, the homotopy totalization of $\mathcal{K}_d^\bullet$ is weakly equivalent to the space $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$ of long knots modulo immersions,*

$$\mathrm{hoTot}\mathcal{K}_d^\bullet \simeq \overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d). \tag{2.3.1}$$

The rest of this section is devoted to an overview of the proof of Theorem 2.3.3. Note that this proof uses in general Goodwillie calculus or the calculus of functors [17, 18], which is a technique that consits of studying a functor F by its "succesive approximations" T_kF. The functor T_kF is called the *kth approximation of* F, and the collection $\{T_kF\}_{k\geq 0}$ is called the *Taylor tower of* F . In the proof of Theorem 2.3.3 we will not use the general theory of Goodwillie , but the Weiss embedding calculus [57] which is more adapted in our case (since the functor with which we are concerned is the compactly supported embedding functor $\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)$).

Let us start with the definition of $\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)$.

Definition 2.3.4.

- *Define $\mathcal{O}^c(\mathbb{R})$ to be the category of open subsets $U \subseteq \mathbb{R}$ on the form $U = \mathbb{R}\backslash K$, where K is a compact subset of I. Morphisms in $\mathcal{O}^c(\mathbb{R})$ are the inclusions.*
- *The contravariant functor*

$$\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)\colon \mathcal{O}^c(\mathbb{R}) \longrightarrow \mathrm{Top}$$

from the category $\mathcal{O}^c(\mathbb{R})$ to topological spaces is defined by

$$\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)(U) = \overline{\mathrm{Emb}}_c(U, \mathbb{R}^d).$$

Let us define now for each $0 \leq k \leq \infty$, the category $\mathcal{O}^c_k(\mathbb{R})$ and the kth approximation $T_k\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)$ of the contravariant functor $\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)$.

Definition 2.3.5. *A family $\{A_i\}_{0\leq i\leq k}$ of subsets of $\mathbb{R}$ is said to be* ordered *if the following two conditions hold*

- *each A_i is a nonempty set;*
- *$x < y$ whenever $x \in A_i, y \in A_j$ and $i < j$.*

This definition implies that elements of an ordered family $\{A_i\}_{0\leq i\leq k}$ are pairwise disjoint.

Definition 2.3.6. *Let $k \geq 0$ be an integer. Define the category $\mathcal{O}^c_k(\mathbb{R})$ (respectively the category $\mathcal{O}^c_\infty(\mathbb{R})$) to be the subcategory of $\mathcal{O}^c(\mathbb{R})$ consisting of open subsets*

$$U = \mathbb{R}\backslash(\cup_{i=0}^p A_i)$$

such that the following two conditions are satisfied

- *p is an integer less than or equal to k (respectively p is any integer);*
- *$\{A_i\}_{0\leq i\leq p}$ is an ordered family of closed subintervals of I.*

The following remark gives an equivalent definition of the category $\mathcal{O}^c_k(\mathbb{R})$.

Remark 2.3.7. *For $0 \leq k \leq \infty$, an open subset $U \subseteq \mathbb{R}$ is an object of the category $\mathcal{O}^c_k(\mathbb{R})$ if and only if it can be written on the form $U = U_0 \cup U_1$, where*

- *U_0 is on the form $U_0 =]-\infty, a[\cup]b, +\infty[$ with $a, b \in I$ and $a < b$;*

- *U_1 is a finite disjoint union of at most k open intervals;*
- *$U_0 \cap U_1 = \emptyset$.*

Notice that the open subset U_0 is just the complement of a closed subinterval of I and this is the reason for which it is called an *anti-interval.* In Chapter 5 (where we will study the space $\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d)$ of "compactly supported" embeddings of an open submanifold $N \subseteq \mathbb{R}^n$ in $\mathbb{R}^d$), the notion of an anti-interval will be generalized by an *anti-ball.*
Consider now the commutative diagram

$$\begin{tikzcd} \mathcal{O}_k^c(\mathbb{R}) \arrow[rr, "\overline{\mathrm{Emb}}_c(-,\mathbb{R}^d)/\mathcal{O}_k^c(\mathbb{R})"] \arrow[d, hook, "i"'] & & \mathrm{Top} \\ \mathcal{O}^c(\mathbb{R}) \arrow[urr, "\overline{\mathrm{Emb}}_c(-,\mathbb{R}^d)"'] & & \end{tikzcd} \tag{2.3.2}$$

Definition 2.3.8. *For $0 \leq k \leq \infty$, the kth approximation*

$$T_k\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d) \colon \mathcal{O}^c(\mathbb{R}) \longrightarrow \mathrm{Top}$$

of the contravariant functor $\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)$ is defined to be the homotopy left Kan extension of the restriction functor $\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)/\mathcal{O}_k^c(\mathbb{R}) \colon \mathcal{O}_k^c(\mathbb{R}) \longrightarrow \mathrm{Top}$ along the inclusion functor $i \colon \mathcal{O}_k^c(\mathbb{R}) \hookrightarrow \mathcal{O}^c(\mathbb{R})$.

By the definition of the homotopy left Kan extension, the kth approximation $T_k\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)$ is explicitly given by the formula

$$T_k\overline{\mathrm{Emb}}_c(U, \mathbb{R}^d) = \underset{V \subseteq U}{\mathrm{holim}}(\overline{\mathrm{Emb}}_c(V, \mathbb{R}^d)), \tag{2.3.3}$$

where the homotopy limit of the right hand side of (2.3.3) runs over open subsets $V \subseteq U$ on the form $V = V_0 \cup V_1$ such that the three conditions of Remark 2.3.7 hold with V_0 and V_1. As a property of the homotopy left Kan extension there exists, for each $0 \leq k \leq \infty$, a natural transformation $\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d) \longrightarrow T_k\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)$ [57]. In [44], Sinha proves that the natural map

$$\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d) \xrightarrow{\sim} T_\infty\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d) = \underset{V \in \mathcal{O}_\infty^c(\mathbb{R})}{\mathrm{holim}} \overline{\mathrm{Emb}}_c(V, \mathbb{R}^d) \tag{2.3.4}$$

is a weak equivalence, when $d \geq 4$.

Before continuing the overview of the proof of Theorem 2.3.3, we recall the notion of a *left cofinal functor.* Let $\theta \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. For $d \in \mathcal{D}$ be an object of $\mathcal{D}$, let $\mathcal{C} \downarrow d$ be the category whose objects are couple (c, f), where c is an object of $\mathcal{C}$ and $f \colon \theta(c) \longrightarrow d$ is a morphism in $\mathcal{D}$. A morphism from (c, f) to (c', f') consists of a morphism $g \colon c \longrightarrow c'$ in $\mathcal{C}$ such that the obvious induced diagram in $\mathcal{D}$ commutes, that is $f'\theta(g) = f$ if θ is a covariant functor and $f\theta(g) = f'$ otherwise.

Definition 2.3.9. *A functor $\theta \colon \mathcal{C} \longrightarrow \mathcal{D}$ is said to be* left cofinal *if for each $d \in \mathcal{D}$, the over category $\mathcal{C} \downarrow d$ is contractible.*

Now we define a contravariant functor $\theta\colon \mathcal{O}^c_\infty(\mathbb{R}) \longrightarrow \Delta$, which turns out to be left cofinal. Recall first the definition of that functor. Let $\{A_i\}_{1\leq i\leq p}$ be a family of ordered closed intervals as above. Set $U = \mathbb{R}\backslash(\cup_{i=0}^p A_i)$. The functor θ is defined on objects by

$$\theta(U) = [p] = \{0, \cdots, p\}.$$

Consider another family $\{B_i\}_{1\leq i\leq q}$ of ordered closed intervals, and set $V = \mathbb{R}\backslash(\cup_{i=0}^q B_i)$. Assume that there is a morphism in $\mathcal{O}^c_\infty(\mathbb{R})$ from U to V, that is $U \subseteq V$. Then for each $i \in [q]$, there exists a unique $j_i \in [p]$ such that $B_i \subseteq A_{j_i}$. Moreover, since the families $\{A_i\}_{1\leq i\leq p}$ and $\{B_i\}_{1\leq i\leq q}$ are ordered, we have $j_{i_1} < j_{i_2}$ whenever $i_1 < i_2$. This defines θ on morphisms by

$$\theta(U \subseteq V)\colon [q] \longrightarrow [p] \qquad \text{with} \qquad \theta(U \subseteq V)(i) = j_i.$$

One can prove the following proposition.

Proposition 2.3.10. *The functor*

$$\theta\colon \mathcal{O}^c_\infty(\mathbb{R}) \longrightarrow \Delta$$

is left cofinal.

Proposition 2.3.11. [45, Proposition 5.15] *Let $U = U_0 \cup U_1$ be an object of the category $\mathcal{O}^c_\infty(\mathbb{R})$ such that U_1 is the disjoint union of exactly k open subintervals of I. Then there is a weak equivalence*

$$\overline{\mathrm{Emb}}_c(U, \mathbb{R}) \simeq \mathrm{Conf}(k, \mathbb{R}^d). \tag{2.3.5}$$

The idea of the proof of Proposition 2.3.11 lies on the fact that one can retract each interval of U_1 to its midpoint.
From weak equivalences (2.2.2) and (2.3.5), it follows that the diagram

$$\begin{tikzcd} \mathcal{O}^c_\infty(\mathbb{R}) \arrow[r, "\overline{\mathrm{Emb}}_c(-{,}\mathbb{R}^d)"] \arrow[d, "\theta"'] & \mathrm{Top} \\ \Delta \arrow[ur, dotted, "\mathcal{K}^\bullet_d"'] & \end{tikzcd}$$

is commutative up to homotopy. In addition the functor θ is left cofinal by Proposition 2.3.10. This implies the following weak equivalence

$$\underset{V\in\mathcal{O}^c_\infty(\mathbb{R})}{\mathrm{holim}}(\overline{\mathrm{Emb}}_c(V, \mathbb{R})) \simeq \mathrm{hoTot}\mathcal{K}^\bullet_d. \tag{2.3.6}$$

Actually the proof of Theorem 2.3.3 follows immediately from (2.3.4) and (2.3.6).

2.4 Collapsing of the H_*BKSS associated to the Sinha cosimplicial model $\mathcal{K}_d^\bullet$

This section recalls the main results of [26] and [27], and makes a comment concerning them. Here $\mathbb{K}$ is a field of characteristic 0.

In [44] Sinha proves that the H_*BKSS associated to $\mathcal{K}_d^\bullet$ converges to the homology $H_*(\mathrm{hoTot}\mathcal{K}_d^\bullet) \cong H_*(\overline{\mathrm{Emb}}_c(\mathbb{R},\mathbb{R}^d))$, when $d \geq 4$. Therefore it is natural to ask whether this spectral sequence collapses or not. This question was studied by Lambrechts, Turchin and Volić in [26], who proved the following result.

Theorem 2.4.1. [26, Theorem 1.2] *For $d \geq 4$ the homology Bousfield-Kan spectral sequence associated to Sinha's cosimplicial space $\mathcal{K}_d^\bullet$ collapses at the E^2 page rationally.*

Definition 2.4.2. *A morphism $f\colon P \longrightarrow Q$ between two topological operads is said to be* $\mathbb{K}$-formal *if there exists, in the category of operads in chain complexes, a zigzag*

$$\begin{tikzcd} S_*(Q;\mathbb{K}) & \bullet \arrow[l, "\sim"'] \arrow[r, "\sim"] & \cdots & \bullet \arrow[l, "\sim"'] \arrow[r, "\sim"] & H_*(Q;\mathbb{K}) \\ S_*(P;\mathbb{K}) \arrow[u, "S_*(f)"] & \bullet \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & \cdots & \bullet \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & H_*(P;\mathbb{K}) \arrow[u, "H_*(f)"'] \end{tikzcd}$$

such that each horizontal arrow is a quasi-isomorphism and each square commutes.

In [27] P. Lambrechts and I. Volić detailed the Kontsevich proof [25] of the $\mathbb{R}$-formality of the little d-disks operad B_d. Moreover, they show another version of this formality: *the relative* $\mathbb{R}$*-formality* (see the following theorem). Notice that the Kontsevich formality and its relative version hold in the category of nonsymmetric operads in chain complexes over $\mathbb{R}$.

Theorem 2.4.3. [27, Theorem 1.4] *For $d \geq 2q+1$ and $q \geq 1$, the inclusion of operads $B_q \hookrightarrow B_d$ is $\mathbb{R}$-formal.*

In the case $q = 1$, the authors of [27] prove Theorem 2.4.3 by building explicitly the following diagram

$$\begin{tikzcd} C_*(\mathcal{K}_d;\mathbb{R}) & C_*(\mathcal{F}_d;\mathbb{R}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathcal{D}_d^\vee & H_*(\mathcal{K}_d;\mathbb{R}) \arrow[l, "\sim"'] \\ C_*(\mathcal{K}_1;\mathbb{R}) \arrow[u] & C_*(\mathcal{F}_1;\mathbb{R}) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & \mathcal{D}_1^\vee \arrow[u] & H_*(\mathcal{K}_1;\mathbb{R}). \arrow[l, "\sim"'] \arrow[u] \end{tikzcd} \tag{2.4.1}$$

In that diagram $C_*(-)$ is a symmetric monoidal functor much like the singular chain functor (further in Section 4.3 we will come back to that functor)

Definition 2.4.4. *Let I be a small category. A diagram $F\colon I \longrightarrow \mathrm{Top}$ is said to be $\mathbb{K}$-formal if $S_*(F;\mathbb{K})$ and $H_*(F;\mathbb{K})$ are weakly equivalent in the category of I-diagrams in chain complexes. In particular, a cosimplicial space $X^\bullet\colon \Delta \longrightarrow \mathrm{Top}$ is $\mathbb{K}$-formal if it is $\mathbb{K}$-formal as a Δ-diagram.*

The following proposition is proved in [26].

Proposition 2.4.5. [26, Proposition 3.2] *Let $X^\bullet \colon \Delta \longrightarrow \mathrm{Top}$ be a $\mathbb{K}$-formal cosimplicial space. Then the homology Bousfield-Kan spectral sequence associated to $X^\bullet$ collapses at the E^2 page rationally.*

Comment In diagram (2.4.1), the operads $C_*\mathcal{K}_d$ and $H_*\mathcal{K}_d$ are multiplicative (by Proposition 1.4.15 and Proposition 2.2.8), and the operad $\mathcal{D}_d^\vee$ is also multiplicative (by Proposition 2.2.22). So we can associate to each of them (by Proposition 1.4.16) a cosimplicial object in chain complexes. If the operad $C_*\mathcal{F}_d$ was multiplicative (unfortunately, as an immediate consequence of Proposition 2.2.8, it isn't), one would deduce the $\mathbb{R}$-formality of the Sinha cosimplicial space $\mathcal{K}_d^\bullet$, and the proof of Theorem 2.4.1 would follow easily (by Proposition 2.4.5). So the problem here is the fact that it is impossible to associate a cosimplicial object, as in the spirit of McClure-Smith (that is, as in Proposition 1.4.16), to the operad $C_*\mathcal{F}_d$. To get around this problem, the authors of [26] introduce certain finite diagrams of spaces called *fanic diagrams*.

In Chapter 4 we will directly prove the $\mathbb{R}$-formality of the Sinha cosimplicial space $\mathcal{K}_d^\bullet$, and therefore this will give us a very short proof (using no fans) of Theorem 2.4.1. Our method will lead us to one of the main results of this thesis: Theorem 4.1.5.

CHAPTER 3

Gerstenhaber algebra structures

3.1 Introduction

This chapter describes Gerstenhaber algebra structures on the Hochschild homology $HH(S_*\mathcal{O})$ associated to a singular chain operad $S_*\mathcal{O}$, on the homology $H_*(\mathrm{Tot}\mathcal{O}^\bullet)$ of the totalization of $\mathcal{O}^\bullet$, and on the homology $H_*(\mathrm{hoTot}\mathcal{O}^\bullet)$ of the homotopy totalization of $\mathcal{O}^\bullet$, when the cosimplicial space $\mathcal{O}^\bullet$ is associated to a multiplicative operad $\mathcal{O}$. In all the chapter $\mathcal{O}^\bullet$ will be always like this (that is, defined by $\mathcal{O}$). We also prove Theorem D (which is the main result of our preprint [48]) announced in the introduction of this thesis.

Outline of the chapter

- In Section 3.2 we first recall the definition of a Gerstenhaber algebra. Next we recall the Hochschild homology $HH(V)$ associated to a multiplicative operad V in graded vector spaces. Finally we equip $HH(V)$ with the natural Gerstenhaber algebra structure induced by the operad structure of V.

- In Section 3.3 we first recall the cacti operad MS, which acts on $\mathrm{Tot}\mathcal{O}^\bullet$. This action was geometrically defined by P. Salvatore in [41, Theorem 5.4]. His idea is to associate from a cactus, elements $a_1^\bullet, \cdots, a_n^\bullet \in \mathrm{Tot}\mathcal{O}^\bullet$, and $t \in \Delta^k$, a planar tree whose vertices (except the root and the leaves) are labelled by the entries of $\mathcal{O}$. Using now the operad structure of $\mathcal{O}$, he gets an operation $\theta \in \mathcal{O}^k$.
  From the same data as those of Salvatore we explicitly construct θ (without using trees), in the proof of Theorem 3.3.11. Our construction is more combinatorial. In fact, we first associate a word (instead of a labelled tree) on the alphabet $\overline{n} = \{1, \cdots, n\}$. Next, by "suitable cutting" this word, we obtain an explicit formula for θ (many illustrative examples are given). After proving Theorem 3.3.11, we recall the E_2 operad

$\mathcal{D}_2$ (introduced in [32] by McClure and Smith), and its action on Tot$\mathcal{O}^\bullet$. Actually it is isomorphic to the cacti operad (see [41, Proposition 8.2]). We end the section with Theorem 3.3.19, which says that the $\mathcal{D}_2$ and MS actions are compatible in the sense that some squares commute. This is the reason for which we need explicit formulas for θ.

- In Section 3.4 we first recall the action of the circle S^1 on Tot$\mathcal{O}^\bullet$ (it was explicitly defined by Sakai in [38, Section 4.1]). From this action he explicitly defines, without proving the compatibility between the S^1 action and the $\mathcal{D}_2$ action, the Gerstenhaber algebra structure (on the homology $H_*(\mathrm{Tot}\mathcal{O}^\bullet)$) induced by the $\mathcal{D}_2$ action. We prove (via the MS action) this compatibility, and therefore the induced Gerstenhaber algebra structures are the same. We end the section with a result of Sakai [38], which states that there exists an isomorphism between $HH(S_*\mathcal{O})$ and $H_*(\mathrm{Tot}\mathcal{O}^\bullet)$ that respects the Gerstenhaber algebra structure, when $\mathcal{O}^\bullet$ is fibrant. He obtains a similar result (that we also recall in Theorem 3.4.5) [38] in the non-fibrant case. Note that in the next chapter, Theorem 3.4.5 (with $\mathcal{O}$ replaced by the Kontsevich operad) will be one of the ingredients in proving Theorem 4.1.5, which is one of the main results in this work.

3.2 Gerstenhaber algebra structure on the Hochschild homology

In this section we first recall the definition of a Gerstenhaber algebra. Next we define the Hochschild homology $HH(V)$ associated to a multiplicative operad V in graded vector spaces. Finally we show that $HH(V)$ is equipped with a natural Gerstenhaber algebra structure. The ground field here is any field $\mathbb{K}$ of characteristic 0.

3.2.1 Definition of a Gerstenhaber algebra

Let us start with some preliminaries definitions.

Definition 3.2.1. • *A* $\mathbb{Z}$-graded vector space *is a direct sum* $A = \bigoplus_{k \in \mathbb{Z}} A_k$, *where each* A_k *is a* $\mathbb{K}$*-vector space. If* $A = \bigoplus_{k \geq 0} A_k$ *we say that* A *is a* positively graded vector space. *An element* $a \in A_k \subseteq A$ *is of degree* k*. Let us denote that degree by* $|a|$*.*

- *For* $p \in \mathbb{Z}$*, the* p-suspension *of a* $\mathbb{Z}$*-graded vector space* A *is the* $\mathbb{Z}$*-graded vector space denoted by* $s^p A$ *and defined by*

$$(s^p A)_k = A_{k-p}. \tag{3.2.1}$$

- *We say that a linear map* $f\colon A \longrightarrow B$ *between two* $\mathbb{Z}$*-graded vector spaces is of* degree p *if the image under* f *of an element of* A *of degree* n *is an element of* B *of degree* $n+p$*, that is,* $f(A_n) \subseteq B_{n+p}$ *for all* n*. A linear map* $f\colon A \longrightarrow B$ *of degree* 0 *is called a* morphism.

- *Let* $A = \oplus_{k\in\mathbb{Z}} A_k$ *and* $B = \oplus_{k\in\mathbb{Z}} B_k$ *be two* $\mathbb{Z}$*-graded vector spaces. The* tensor product *(respectively the* direct sum*) of* A *and* B *is the* $\mathbb{Z}$*-graded vector space denoted by* $A \otimes B$ *(respectively by* $A \oplus B$*) and defined by*

$$(A \otimes B)_n = \bigoplus_{p+q=n} A_p \otimes B_q \ \textit{(respectively by } (A \oplus B)_n = A_n \oplus B_n\textit{)}.$$

The p-suspension of an element $a \in A$ will be denoted by $s^p a$. Notice that if for instance $a \in A$ is of degree $p+k$, then $|s^{-p}a| = k$ (this comes from (3.2.1) above).

Definition 3.2.2. *A* Poisson algebra of degree r *or a* r-Poisson algebra *(for* $r > 1$*) is a triple* $(A, \times, \{-,-\})$*, where*

- $A = \bigoplus_{k\in\mathbb{Z}} A_k$ *is a* $\mathbb{Z}$*-graded vector space,*

- $\times \colon A \otimes A \longrightarrow A$ *is an associative bilinear operation of degree* 0*, called the* product*,*

- $\{-,-\} \colon A \otimes A \longrightarrow A$ *is a bilinear operation of degree* $r-1$*, called the* bracket*,*

such that we have the

- *graded commutativity of the product,*

$$a \times b = (-1)^{|a||b|} b \times a;$$

- *graded antisymmetry relation,*

$$\{a, b\} = -(-1)^{(|a|+1-r)(|b|+1-r)}\{b, a\};$$

- *graded Jacobi identity,*

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1-r)(|b|+1-r)}\{b, \{a, c\}\};$$

- *biderivative of the bracket with respect to the product,*

$$\{a, b \times c\} = \{a, b\} \times c + (-1)^{|b|(|a|+1-r)} b \times \{a, c\}.$$

The product $a \times b$ will be denoted just by ab.

A *Gerstenhaber algebra* is a particular case of a Poisson algebra. More precisely, we have the following definition.

Definition 3.2.3. *A* Gerstenhaber algebra *is a Poisson algebra of degree* 2*.*

3.2.2 Hochschild homology of a multiplicative operad in graded vector spaces

Let $V = \{V(k)\}_{k\geq 0}$ be a multiplicative operad in graded vector spaces (the notion of a multiplicative operad was introduced in Definition 1.4.13). In Proposition 1.4.16, we have seen the construction that associates a cosimplicial object to any multiplicative operad. Taking V as input in that construction, we get the cosimplicial object $V^\bullet$. Recall that V^k is defined by $V^k = V(k)$. Define a morphism $d\colon V(k) \longrightarrow V(k+1)$ by

$$d = \sum_{i=0}^{k+1} (-1)^i d^i, \tag{3.2.2}$$

where (for $0 \leq i \leq k+1$), $d^i\colon V(k) \longrightarrow V(k+1)$ is the ith coface morphism of $V^\bullet$. Using the cosimplicial relations (from (1.2.8) of Chapter 1) one can show that d is a differential. That is,

$$d^2 = 0. \tag{3.2.3}$$

We will define two complexes associated to V: a chain complex $C_*H(V)$ and a cochain complex $C^*H(V)$. Let us start with $C^*H(V)$. Since each $V(k)$ is a graded vector space, we can set $V(k) = \underset{i\in\mathbb{Z}}{\oplus} V(k)_i$. First define a vector space

$$C^n H(V) = \bigoplus_{k\geq 0} (s^n V(k))_k.$$

Next define $C^*H(V)$ by

$$C^*H(V) = \bigoplus_{n\geq 0} C^n H(V).$$

Finally define $\delta\colon C^nH(V) \longrightarrow C^{n+1}H(V)$ by

$$\delta(\underset{k\geq 0}{\oplus} x_k) = \underset{k\geq 0}{\oplus} d(x_k).$$

It is clear that δ is a differential since $d^2 = 0$ from (3.2.3). We then form the cochain complex $(C^*H(V), \delta)$, which will be called the *Hochschild cocomplex* associated to V. We call it like this because, under good conditions, it is related to the classical Hochschild cocomplex as we will see now. Let A be an associative algebra. As usual let $C^*(A, A)$ denote the classical Hochschild cocomplex. Recall that

$$C^n(A, A) = \mathrm{Hom}(A^{\otimes n}, A), \qquad \text{where} \qquad A^{\otimes n} = \underbrace{A \otimes \cdots \otimes A}_{n}.$$

The differential

$$\widetilde{\delta}\colon C^n(A, A) \longrightarrow C^{n+1}(A, A) \text{ is defined by } \widetilde{\delta}(f) = \sum_{i=0}^{n+1} (-1)^i \widetilde{d^i}(f),$$

where for every $a = (a_1, \cdots, a_{n+1}) \in A^{\otimes(n+1)}$,

$$\widetilde{d^i}(f)(a) = \begin{cases} a_1 f(a_2, \cdots, a_{n+1}) & \text{if} \quad i = 0 \\ f(a_1, \cdots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \cdots, a_{n+1}) & \text{if} \quad 1 \le i \le n \\ f(a_1, \cdots, a_n) a_{n+1} & \text{if} \quad i = n+1. \end{cases}$$

Consider now the endomorphism operad End_A (it was defined in Example 1.4.7 from Chapter 1). This operad turns out to be a multiplicative operad in vector spaces, the special operation in $End_A(n) = \mathrm{Hom}(A^{\otimes_n}, A)$ being induced by the multiplication in A. This makes sense because of the associativity of A. Taking End_A as input in the previous construction, we get the cochain complex $C^*H(End_A)$. The following proposition, which is very easy to prove, says that this latter complex is nothing other than the classical Hochschild cocomplex associated to A.

Proposition 3.2.4. *Let A be an associative algebra. Then we have the equality*

$$C^*H(End_A) = C^*(A, A).$$

Let us define now $C_*H(V)$ (we will denote it just by $CH(V)$), which is roughly speaking the dual of $C^*H(V)$. Set

$$CH(V)_n = \bigoplus_{k \ge 0} (s^{-k}V(k))_n,$$

and define $CH(V) = \bigoplus\limits_{n \ge 0} (CH(V))_n$. We want to endow $CH(V)$ with a differential $\partial\colon CH(V)_n \longrightarrow CH(V)_{n-1}$. Let $x = \underset{k \ge 0}{\oplus} x_k$ be an element of $CH(V)_n$. Then each $x_k \in (s^{-k}V(k))_n = V(k)_{n+k}$ (by (3.2.1) above). Applying the differential d to each x_k, we have the element

$$d(x_k) \in V(k+1)_{n+k} = (s^{-(k+1)}V(k+1))_{n-1}.$$

So $\partial(x) = \underset{k \ge 0}{\oplus} d(x_k)$ is an element of $CH(V)_{n-1}$. This defines ∂ by

$$\partial(\underset{k \ge 0}{\oplus} x_k) = \underset{k \ge 0}{\oplus} d(x_k). \tag{3.2.4}$$

Proposition 3.2.5. *The morphism $\partial\colon CH(V)_n \longrightarrow CH(V)_{n-1}$ is a differential.*

Proof. Let $x = \underset{k \ge 0}{\oplus} x_k$ be an element of $CH(V)_n$. Then

$$\begin{aligned} \partial^2(x) &= \partial(\underset{k \ge 0}{\oplus} d(x_k)) \text{ by (3.2.4)} \\ &= \underset{k \ge 0}{\oplus} d^2(x_k) \text{ by (3.2.4)} \\ &= 0 \text{ since } d^2 = 0 \text{ by (3.2.3).} \end{aligned}$$

□

Definition 3.2.6. *Consider the chain complex* $(CH(V), \partial)$ *we just defined.*

- $(CH(V), \partial)$ *is called the* Hochschild complex *associated to the multiplicative operad* V*.*
- *The homology of the Hochschild complex is called the* Hochschild homology of V*, and it is denoted by* $HH(V)$*.*

The Hochschild homology $HH(V)$ is equipped with a natural Gerstenhaber algebra structure induced by the multiplicative structure of the operad V. To be precise, define on the Hochschild complex $CH(V)$ a product $\times$ and a bracket $\{-,-\}$ as follows. Let $\mu \in V(2)$ denote the special operation in arity 2 (such an operation exists because V is multiplicative). For $a \in (s^{-p}V(p))_i = V(p)_{i+p}$ and $b \in (s^{-q}V(q))_j = V(q)_{j+q}$, define $a \times b$ by

$$a \times b = (\mu \circ_2 b) \circ_1 a = \mu(a, b), \tag{3.2.5}$$

and define $\{a, b\}$ by

$$\{a, b\} = \sum_{r=1}^{p} (-1)^{\epsilon} a \circ_r b - (-1)^{(i+1)(j+1)} \sum_{t=1}^{q} (-1)^{\epsilon'} b \circ_t a, \tag{3.2.6}$$

where

$$\epsilon = (i-1)(p-r) + (p-1)(q+j) \qquad \text{and} \qquad \epsilon' = (p-1)(q-t) + (p+i)(q-1).$$

Here $\circ_i$ is the ith insertion morphism of the operad structure of V. Notice that the product $a \times b$ sits in

$$V(p+q)_{i+j+p+q} = (s^{-(p+q)}V(p+q))_{i+j},$$

while the bracket $\{a, b\}$ sits in

$$V(p+q-1)_{i+j+p+q} = (s^{-(p+q-1)}V(p+q-1))_{i+j+1}.$$

So the formulas (3.2.5) and (3.2.6) respectively induce a linear map

$$\times \colon CH(V) \otimes CH(V) \longrightarrow CH(V) \qquad \text{of degree } 0,$$

and a linear map

$$\{-,-\} \colon CH(V) \otimes CH(V) \longrightarrow CH(V) \qquad \text{of degree } 1.$$

Notice that the differential ∂ (defined by the formula (3.2.4) above) of the chain complex $CH(V)$ can be redefined in term of the bracket. More exactly, we have

$$\partial(x) = \{\mu, x\}.$$

Proposition 3.2.7. [14] *The product (3.2.5) and the bracket (3.2.6) induce a Gerstenhaber algebra structure on the Hochschild homology* $HH(V)$*.*

3.3 The cacti operad MS, the McClure-Smith operad $\mathcal{D}_2$ and their actions on Tot$\mathcal{O}^\bullet$

In this section we first define the cacti operad MS (see Definition 3.3.9) and show that it explicitly acts on Tot$\mathcal{O}^\bullet$ (Theorem 3.3.11), where $\mathcal{O}^\bullet$ is a cosimplicial space defined by a multiplicative operad $\mathcal{O}$. Next we recall the definition of the operad $\mathcal{D}_2$ (built by McClure-Smith in [32]) and its action on Tot$\mathcal{O}^\bullet$. We end with Theorem 3.3.19, which states that the MS and $\mathcal{D}_2$ actions are compatible. In all this section, for $n \geq 0$, the set $\overline{n}$ is defined by $\overline{n} = \{1, \cdots, n\}$.

3.3.1 The cacti operad MS and its action on Tot$\mathcal{O}^\bullet$

We begin with the definition of the cacti operad MS. Let S^1 be the unit circle viewed as the quotient of the interval $I = [-1, 1]$ by the relation $-1 \sim 1$. Let $\pi \colon [-1, 1] \longrightarrow S^1$ denote the canonical surjection, and let $* = \pi(1)$ denote the base point of S^1. To define MS we need to define first a family of topological spaces

$$\mathcal{I} = \{\mathcal{I}_k(n) : n \geq 0 \text{ and } 0 \leq k \leq \infty\}.$$

Let $n \geq 1$ be an integer. We start by defining the space $\mathcal{I}_\infty(n)$. Next we will define $\mathcal{I}_k(n)$ as a subspace of $\mathcal{I}_\infty(n)$.
Let $K = \{K_i = [x_i^K, x_{i+1}^K]\}_{i=0}^{p_K - 1}$ be a family of closed subintervals of I satisfying the following two conditions:

(P_1): $p_K \geq n$, $x_0^K = -1$, $x_p^K = 1$ and the points $x_0^K, \cdots, x_p^K$ are pairwise distinct.

(P_2): The intervals K_i define n 1-manifolds $I_1^K, \cdots, I_n^K$ of disjoint interiors and with equal length. This means that each I_j^K is an union of some K_i.

We denote by $\mathcal{P}_n$ the collection of such a family K. That is,

$$\mathcal{P}_n = \{K = \{K_i = [x_i^K, x_{i+1}^K]\}_{i=0}^{p_K-1} | \ K \text{ satisfies } (P_1) \text{ and } (P_2)\}. \quad (3.3.1)$$

The image of a n-tuple $(I_1^K, \cdots, I_n^K)$ (respectively the image of intervals K_i) under the canonical surjection π will be denoted again by $(I_1^K, \cdots, I_n^K)$ (respectively by $\{K_i\}$). The set $\mathcal{I}_\infty(n)$ is then defined by

$$\mathcal{I}_\infty(n) = \{(I_1^K, \cdots, I_n^K) | \ K \in \mathcal{P}_n\}.$$

From now and in the rest of this section, we will denote an element $x \in \mathcal{I}_\infty(n)$ by $x = (I_1(x), \cdots, I_n(x))$ or just by $x = (I_1, \cdots, I_n)$. The family $K = \{K_i = [x_i^K, x_{i+1}^K]\}_{i=0}^{p_K-1}$ will be sometimes just denoted by $K = \{K_i = [x_i, x_{i+1}]\}_{i=0}^{p-1}$.
Let us equip now the set $\mathcal{I}_\infty(n)$ with the following topology. Two elements

$$x = (I_1(x), \cdots, I_n(x)) \qquad \text{and} \qquad y = (I_1(y), \cdots, I_n(y))$$

of $\mathcal{I}_\infty(n)$ are said to be close if for each i there is $\epsilon_i > 0$ too small such that

$$\text{length}\,((I_i(x) \cup I_i(y)) \backslash (I_i(x) \cap I_i(y))) < \epsilon_i.$$

Notice that $\mathcal{I}_\infty(1)$ is the one point space. In order to define the space $\mathcal{I}_k(n)$ (for $k \geq 0$ be an integer), recall first the notion of the *complexity* of a map.

Definition 3.3.1. *Let T be a finite totally ordered set, n be an integer, and $f\colon T \longrightarrow \overline{n} = \{1, \cdots, n\}$ be a map. The* complexity *of f, denoted by* $\mathrm{cplx}(f)$*, is defined as follows.*

- *If $n = 0$ or $n = 1$ then* $\mathrm{cplx}(f) = 0$.
- *If $n = 2$, let $\sim$ be the equivalence relation on T generated by*

$$a \sim b \text{ if } a \text{ is adjacent to } b \text{ and } f(a) = f(b).$$

  The complexity of f is equal to the number of equivalence classes minus 1.

- *If $n > 2$, let $f_{ij}\colon f^{-1}(\{i, j\}) \longrightarrow \{i, j\}$ denote the restriction of f on $f^{-1}(\{i, j\})$. The complexity of f is equal to the maximum of complexities of the restrictions f_{ij} as $\{i, j\}$ ranges over the two-element subsets of $\overline{n}$. That is,*

$$\mathrm{cplx}(f) = \underset{1 \leq i < j \leq n}{\mathrm{Max}}(\mathrm{cplx}(f_{ij})). \tag{3.3.2}$$

Note that a map $f\colon T \longrightarrow \{1, \cdots, n\}$ can be viewed as a word of length $|T|$ on the alphabet $\{1, \cdots, n\}$ (here $|T|$ denotes the cardinal of T). The *length of the alphabet* $\{1, \cdots n\}$ is defined to be its cardinal.

Example 3.3.2. *(a) If $|T| = 5$, $n = 2$ and f is defined by the word $f = 12212$ then* $\mathrm{cplx}(f) = 3$.

(b) Assume that $|T| = 8$, $n = 3$ and $f = 31232113$, and consider the following tabular

map	$f_{12} = 12211$	$f_{13} = 313113$	$f_{23} = 32323$
complexity	$\mathrm{cplx}(f_{12}) = 2$	$\mathrm{cplx}(f_{13}) = 4$	$\mathrm{cplx}(f_{23}) = 4$

By this tabular and by (3.3.2), we deduce that $\mathrm{cplx}(f) = 4$.

Let $x = (I_1, \cdots, I_n) \in \mathcal{I}_\infty(n)$ defined by a partition $K = \{K_i\}_{i=0}^{p-1} \in \mathcal{P}_n$. Let T_x be the set defined by $T_x = \{0, 1, \cdots, p-1\}$. For each $k \in T_x$, it is clear that there exixts a unique $i_k \in \{1, \cdots, n\}$ such that $K_k \subseteq I_{i_k}$ (this comes from the condition (P_2) above). This defines a map

$$f_x\colon T_x \longrightarrow \{1, \cdots, n\} \tag{3.3.3}$$

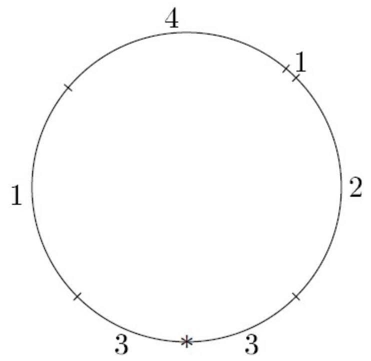
Definition 3.3.3. *The* complexity *of an element $x \in \mathcal{I}_\infty(n)$, denoted by* $\mathrm{cplx}(x)$*, is defined to be the complexity of the map f_x. That is,*

$$\mathrm{cplx}(x) = \mathrm{cplx}(f_x).$$

We are now ready to define $\mathcal{I}_k(n)$.

Definition 3.3.4. *For $k \geq 0$, the space $\mathcal{I}_k(n)$ is the subspace of $\mathcal{I}_\infty(n)$ defined by*

$$\mathcal{I}_k(n) = \{x \in \mathcal{I}_\infty(n) |\ \mathrm{cplx}(x) \leq k\}.$$



The picture above is an element of $\mathcal{I}_2(4)$.

Remark 3.3.5. *The space $\mathcal{I}_2(n)$ is a finite regular CW-complex with one cell for each $f_x(T_x)$ (see (3.3.3) above for the definition of f_x). A cell labelled by some $f_x(T_x)$ is homeomorphic to $\prod_{i=1}^{n} \Delta^{|f_x^{-1}(i)|-1}$. For instance, the element of the picture above is an element of a cell homeomorphic to $\Delta^1 \times \Delta^1$, which is labelled by* 314123*.*

Now we want to define a *cactus with n lobes.* Let $x = (I_1, \cdots, I_n) \in \mathcal{I}_2(n)$ defined by a family of closed intervals $K = \{K_{i1}, \cdots, K_{il_i}\}_{i=1}^{q} \in \mathcal{P}_n$. Assume that each I_i is on the form $I_i = \cup_{r=1}^{l_i} K_{ir}$, and define on S^1 the equivalence relation $\sim_i$ (for $1 \leq i \leq n$) generated by

$$(t_1 \sim_i t_2) \text{ if and only if } (t_1, t_2 \in K_{jr} \text{ with } jr \notin \{i1, \cdots, il_i\}).$$

It is easy to see that the quotient of S^1 by this equivalence is homeomorphic to S^1. Let us denote by π_i the canonical surjection.

$$\pi_i \colon S^1 \longrightarrow S^1/\sim_i \cong S^1.$$

We thus construct a map

$$c(x) \colon S^1 \longrightarrow (S^1)^n$$

defined by $c(x) = (\pi_1, \cdots, \pi_n)$, and called the *cactus map.* Its image is called the *cactus with n lobes* associated to $x \in \mathcal{I}_2(n)$. Recalling the definition of the coendomorphism operad Coend_{S^1} from Example 1.4.9, there is an embedding

$$\overline{\tau}_n \colon \mathcal{I}_2(n) \longrightarrow \mathrm{Coend}_{S^1}(n)$$

defined by

$$\overline{\tau}(x) = c(x).$$

Remark 3.3.6. *The collection of spaces $\{\overline{\tau}_n(\mathcal{I}_2(n))\}_{n\geq 0}$ is not a suboperad of $\mathrm{Coend}_{S^1}(\bullet)$. Indeed, let x be an element of $\mathcal{I}_2(2)$ laballed by* 212*. Then*

$c(x) = (\pi_1, \pi_2) \in \mathrm{Coend}_{S^1}(2)$. *Using now the operad structure of* Coend_{S^1}, *we get*

$$c(x) \circ_2 c(x) = (\pi_1, \pi_1 \circ \pi_2, \pi_2 \circ \pi_2) = (\pi_1, \textit{constant map}, \pi_2 \circ \pi_2),$$

and it is impossible to find an element $z \in \mathcal{F}_2(3)$ *such that*

$$c(z) = (\pi_1, \textit{constant map}, \pi_2 \circ \pi_2).$$

Since the collection $\{\overline{\tau}_n(\mathcal{F}_2(n))\}_{n\geq 0}$ is not far to be an operad, to get the right one, we introduce the space $\mathrm{Mon}(I, \partial I)$ defined as follows. Let $\partial I = \{-1, 1\}$ denotes the boundary of I. Let $\varphi\colon I \longrightarrow I$ be a weakly monotone map such that its restriction on ∂I coincides with the identity map $id_{\partial I}$. Then the map φ passes to the quotient and gives a map $\widetilde{\varphi}\colon S^1 \longrightarrow S^1$, which is a typical element of $\mathrm{Mon}(I, \partial I)$.

Remark 3.3.7. *The space* $\mathrm{Mon}(I, \partial I)$ *is homeomorphic to the totalization* $\mathrm{Tot}\Delta^\bullet \simeq *$. *The homeomorphism* $\mathrm{Mon}(I, \partial I) \xrightarrow{\cong} \mathrm{Tot}\Delta^\bullet$ *sends* $\widetilde{\varphi} \in \mathrm{Mon}(I, \partial I)$ *to* $(t_1, \cdots, t_k) \longmapsto (\varphi(t_1), \cdots, \varphi(t_k))$.

Considering the embedding

$$\tau_n\colon \mathcal{I}_2(n) \times \mathrm{Mon}(I, \partial I) \longrightarrow \mathrm{Coend}_{S^1}(n)$$

defined by

$$\tau_n(x, \widetilde{\varphi}) = c(x) \circ \widetilde{\varphi}\colon S^1 \longrightarrow (S^1)^n,$$

we have the following proposition.

Proposition 3.3.8. *The collection*

$$\{\mathrm{im}(\tau_n)\}_{n\geq 0} = \{\tau_n(\mathcal{I}_2(n) \times \mathit{Mon}(I, \partial I))\}_{n\geq 0}$$

is a suboperad of Coend_{S^1}.

Proof. The proof follows immediately from [41, Proposition 4.5]. □

Let $MS = \{MS(n)\}_{n\geq 0}$ be the collection of topological spaces defined by

$$MS(n) = \begin{cases} \mathcal{I}_2(n) \times \mathrm{Mon}(I, \partial I) & \text{if} \quad n \geq 1 \\ * & \text{if} \quad n = 0. \end{cases} \tag{3.3.4}$$

By transferring the operad structure of $\{\mathrm{im}(\tau_n)\}_{n\geq 0}$ (which is given by Proposition 3.3.8) on MS via embeddings τ_n, we endow MS with an operad structure.

Definition 3.3.9. *The operad* MS *is called the cacti operad.*

Proposition 3.3.10. *There exists a weak equivalence* $(\phi, id)\colon S^1 \xrightarrow{\sim} MS(2)$.

Proof. Define first intervals K_0, K_1 and K_2 by

$$\begin{cases} K_0=[-1,\tau], K_1=[\tau,1+\tau], K_2=[1+\tau,1] & \text{if} \quad -1<\tau<0 \\ K_0=[-1,\tau-1], K_1=[\tau-1,\tau], K_2=[\tau,1] & \text{if} \quad 0<\tau<1 \\ K_0=[-1,0], K_1=[0,1] & \text{if} \quad \tau=0 \\ K_0=[-1,0], K_1=[0,1] & \text{if} \quad \tau=\pm1. \end{cases}$$

Define next a map $\phi\colon S^1 \longrightarrow \mathcal{I}_2(2)$ by

$$\phi(\tau)=\begin{cases} (I_1=K_0\cup K_2, I_2=K_1) & \text{if} \quad -1<\tau<0 \\ (I_1=K_1, I_2=K_0\cup K_2) & \text{if} \quad 0<\tau<1 \\ (I_1=K_0, I_2=K_1) & \text{if} \quad \tau=0 \\ (I_1=K_1, I_2=K_0) & \text{if} \quad \tau=\pm1. \end{cases} \tag{3.3.5}$$

It is not difficult to see that ϕ is a homeomorphism. Therefore the map

$$(\phi, id)\colon S^1 \longrightarrow MS(2)=\mathcal{I}_2(2)\times \mathrm{Mon}(I,\partial I), \tag{3.3.6}$$

where id is the map that sends each point of S^1 to the identity map $id_{S^1}: S^1 \longrightarrow S^1$, is a weak equivalence since the space $\mathrm{Mon}(I,\partial I)$ is contractible by Remark 3.3.7. □

The following theorem is originally due to Salvatore in [41]. He gives a nice geometric proof to it. Here we give another proof, which is more combinatorial. We provide in fact explicit formulas of the action of MS on Tot$\mathcal{O}^\bullet$.

Theorem 3.3.11. [41, Theorem 5.4] *Let $\mathcal{O}^\bullet$ be a cosimplicial space defined by a multiplicative operad $\mathcal{O}$. Then the cacti operad MS acts on the totalization* Tot$\mathcal{O}^\bullet$.

Proof. Let $(x,\widetilde{\varphi})\in MS(n)=\mathcal{I}_2(n)\times \mathrm{Mon}(I,\partial I)$, let $a_i^\bullet \in \mathrm{Tot}\mathcal{O}^\bullet$, $1\leq i\leq n$. Our aim is to construct from these data a family

$$\theta_n((x,\widetilde{\varphi}),(a_1^\bullet,\cdots,a_n^\bullet))_k\colon \Delta^k \longrightarrow \mathcal{O}^k, k\geq 0, \tag{3.3.7}$$

of maps that commute with cofaces and codegeneracies.

Let $t=(-1=t_0\leq t_1\leq\cdots\leq t_k\leq t_{k+1}=1)$ be an element of Δ^k. Define a family of closed intervals $\{J_j\}_{j=0}^k$ by setting $J_j=[\varphi(t_j),\varphi(t_{j+1})]$. Assume that $x\in\mathcal{I}_2(n)$ is defined by a family of p closed intervals $K=\{[x_i,x_{i+1}]\}_{i=0}^{p-1}$, and consider the set

$$E=\{\varphi(t_j)|\ 0\leq j\leq k+1\}\cup\{x_i|\ 0\leq i\leq p\}.$$

Define now a family $\{L_l=[a_l,b_l]\}_{l=0}^m$ of closed subintervals of I as follows.

- each a_l or b_l belongs to E;
- the interior of each L_l contains no element of E;
- $\cup_{l=0}^m L_l=I$ and

$$m=k+p-1. \tag{3.3.8}$$

(a) Assume that for all i and for all j, $x_i \neq \varphi(t_j)$

In this case there exists, for each $l \in \{0, \cdots, m\}$, an unique element $i_l \in \{1, \cdots, n\}$ and an unique element $j_l \in \{0, \cdots, k\}$ such that $L_l \subseteq I_{i_l}$ and $L_l \subseteq J_{j_l}$. This gives two maps

$$[k] \xleftarrow{h} [m] \xrightarrow{f} \overline{n} \tag{3.3.9}$$

defined by

$$h(l) = j_l \qquad \text{and} \qquad f(l) = i_l.$$

It is easy to see that the map h is a morphism in the simplicial category Δ. It is also easy to see that f is a surjective map, and its complexity is less than or equal to 2 (this is because we have taken x in $\mathcal{I}_2(n)$, and by Definition 3.3.4 we have cplx(x) ≤ 2).

Let $i \in \overline{n}$. We want to explicitly construct an element $y_i \in \Delta^{f^{-1}(i)} = \Delta^{|f^{-1}(i)|-1}$. Let us set

$$I_i = \cup_{j=0}^{l_i} K_{i_j}, K_{i_j} = [x_{i_j}, x_{i_j+1}] \subseteq [-1,1], \text{and } x_{i_{l_i+1}} = 1.$$

Define by induction a family $\{g_{i_q} \colon [-1,1] \longrightarrow [-1,1]\}_{q=0}^{l_i+1}$ of maps as follows.

$$g_{i_0}(z) = \begin{cases} -1 & \text{if} \quad z \in [-1, x_{i_0}] \\ z - (x_{i_0} + 1) & \text{if} \quad z > x_{i_0}. \end{cases} \tag{3.3.10}$$

For $0 \leq q \leq l_i$, set $g_q = g_{i_q} \circ \cdots \circ g_{i_0}$ and define

$$g_{i_{q+1}}(z) = \begin{cases} z & \text{if} \quad z \in [-1, g_q(x_{i_q+1})] \\ g_q(x_{i_q+1}) & \text{if} \quad z \in [g_q(x_{i_q+1}), g_q(x_{i_{q+1}})] \\ z - (x_{i_{q+1}} - x_{i_q+1}) & \text{if} \quad z > g_q(x_{i_{q+1}}). \end{cases} \tag{3.3.11}$$

Intuitively, the map g_{i_0} contracts the interval $[-1, x_{i_0}]$ to -1, and moves other points by the translation of vector $-(x_{i_0}+1)$, the map g_{i_1} contracts the interval $[g_{i_0}(x_{i_0+1}), g_{i_0}(x_{i_1})]$ to $g_{i_0}(x_{i_0+1})$, and moves other points by the translation of vector $-(x_{i_1} - x_{i_0+1})$, and so on. At the end of this process, we obtain an interval of length $\frac{2}{n}$. More precisely, if we define g to be the composite

$$g = g_{i_{l_i+1}} \circ g_{i_{l_i}} \circ \cdots \circ g_{i_1} \circ g_{i_0}, \tag{3.3.12}$$

then

$$g([-1,1]) = [-1, \frac{2-n}{n}].$$

Define also a map $\alpha \colon [-1, \frac{2-n}{n}] \longrightarrow [-1,1]$ by

$$\alpha(z) = nz + n - 1. \tag{3.3.13}$$

Notice the map α fixes -1, and sends $\frac{2-n}{n}$ to 1. In fact α allows to rescale the interval $[-1, \frac{2-n}{n}]$. Consider now the map $\widetilde{g} \colon [-1,1] \longrightarrow [-1,1]$ defined by

$$\widetilde{g} = \alpha \circ g.$$

For $j \in \{0, \cdots, l_i\}$, if $t^1_{i_j}, \cdots, t^{v_j}_{i_j}$ denote elements of the set $\{\varphi(t_1), \cdots, \varphi(t_k)\}$ that belong to K_{i_j}, then $y_i \in \Delta^{|f^{-1}(i)|-1}$ is defined by

$$y_i = (\widetilde{g}(t^1_{i_0}), \cdots, \widetilde{g}(t^{v_0}_{i_0}), \widetilde{g}(x_{i_0+1})), \cdots, \widetilde{g}(t^1_{i_{l_i}}), \cdots, \widetilde{g}(t^{v_{l_i}}_{i_{l_i}})). \tag{3.3.14}$$

An illustration for y_i is given in the first part of Example 3.3.12.

Let us construct now by induction on n the operation

$$\theta_n(((x, \widetilde{\varphi}), (a^\bullet_1, \cdots, a^\bullet_n))_k(t) \in \mathcal{O}(k).$$

The map f will be thought as a word of length $m+1$ on the alphabet $\{1, \cdots, n\}$. If W is a word on an alphabet of length $*$, we will write $\theta_*(W)$ for the associated operation. For instance, the operation $\theta_n(((x, \widetilde{\varphi}), (a^\bullet_1, \cdots, a^\bullet_n))_k(t)$ will be sometimes denoted by $\theta_n(f)$.

If $n = 1$ then $c(x) = id_{S^1}$. Define $\theta_1((x, \widetilde{\varphi}), a^\bullet_1)(t) \in \mathcal{O}(k)$ by

$$\theta_1((x, \widetilde{\varphi}), a^\bullet_1)(t) = a^k_1(t)$$

If $n = 2$, let i, j be two distinct elements inside $\{1, 2\}$. Since the complexity of f is less than or equal to 2, there are two possibilities for writing the word f.

- Assume that the map $f\colon [m] \longrightarrow \{1, 2\}$ is on the form $f = \underbrace{i \cdots i}_{r+1}\underbrace{j \cdots j}_{s+1}$.

  This implies that we have exactly two closed intervals K_0, K_1 defining $x = (I_1, I_2) \in \mathcal{I}_2(2)$, and therefore, $p = 2$ (recall that p is the number of intervals K_i defining $x \in \mathcal{I}_2(2)$). We claim that $r + s = k$. Indeed, since the length of the word f is equal to $m + 1$, it follows that

$$\begin{array}{rcl} (r+1) + (s+1) & = & m+1 \\ & = & k+p \quad \text{since } m = k+p-1 \text{ by (3.3.8) above} \\ & = & k+2 \quad \text{since } p = 2. \end{array}$$

  Let $\mu \in \mathcal{O}(2)$ denote the multiplication. Define $\theta_2((x, \widetilde{\varphi}), (a^\bullet_1, a^\bullet_2))_k(t) \in \mathcal{O}(r+s) = \mathcal{O}(k)$ by the formula

$$\theta_2(f) = \theta_2(i \cdots ij \cdots j) = \mu(a^r_i(y_i), a^s_j(y_j)). \tag{3.3.15}$$

- Now we assume that the word f is on the form $f = \underbrace{i \cdots i}_{r_1}\underbrace{j \cdots j}_{s+1}\underbrace{i \cdots i}_{r_2}$.

  This implies that $p = 3$. Like before, we can check that $r_1 + r_2 - 1 + s - 1 = k$. Define the operation

$$\theta_2((x, \widetilde{\varphi}), (a^\bullet_1, a^\bullet_2))_k(t) \in \mathcal{O}(r_1 + r_2 - 1 + s - 1) = \mathcal{O}(k)$$

  by

$$\theta_2(f) = \theta_2(i \cdots ij \cdots ji \cdots i) = a^{r_1+r_2-1}_i(y_i) \circ_{r_1} a^s_j(y_j). \tag{3.3.16}$$

Let $n \geq 3$. For $i \leq n-1$, the operation $\theta_i((x, \widetilde{\varphi}), (a_1^\bullet, \cdots, a_i^\bullet))_k$ will be denoted just by $\theta_*(-)$. Assume that $\theta_*(W)$ is constructed for each word W on an alphabet of length $* \leq n-1$. We want to construct $\theta_n(f)$. Set $f(0) = i_0$ and define the integer

$$m_0 = \mathrm{Max}\{j \in [m] |\ f(j) = i_0\}.$$

There are two possibilities depending of the fact that the word f ends by the letter i_0 or not.

If $m_0 = m$ then the map f is on the form

$$f = \underbrace{i_0 \cdots i_0}_{r_1} b_{11} \cdots b_{1s_1} \underbrace{i_0 \cdots i_0}_{r_2} b_{21} \cdots b_{2s_2} \cdots \underbrace{i_0 \cdots i_0}_{r_q} b_{q1} \cdots b_{qs_q} \underbrace{i_0 \cdots i_0}_{r_{q+1}},$$

with $b_{js} \neq i_0$ for all j and for all s, and with $r_1 + \cdots + r_{q+1}$ copies of i_0. Let us set

$$u = \left(\sum_{i=1}^{q+1} r_i\right) - 1 \qquad \text{and} \qquad v = \sum_{i=1}^{q} r_i.$$

Define $\theta_n(f)$ by the formula

$$\theta_n(f) = (((a_{i_0}^u(y_{i_0}) \circ_v \theta_*(b_{q1} \cdots b_{qs_q})) \circ_{v-r_q} \cdots) \circ_{r_1} \theta_*(b_{11} \cdots b_{1s_1}). \quad (3.3.17)$$

A perfect illustration for this formula is given by Example 3.3.13 below.

If $m_0 < m$ then the map f is on the form

$$f = i_0 \cdots i_0 b_1 \cdots b_w i_0 \cdots i_0 c_1 \cdots c_s$$

with $c_j \neq i_0$ for all j. Let r be the number of letters (in the alphabet $\{1, \cdots, n\}$) appearing in the word

$$f(\{0, \cdots, m_0\}) = i_0 \cdots i_0 b_1 \cdots b_w i_0 \cdots i_0.$$

Then, since each letter of the word $b_1 \cdots b_w$ does not appear in the word $c_1 \cdots c_s$ (because the complexity of f is less than or equal to 2), there is exactly $n-r$ letters appearing in the word $c_1 \cdots c_s$. Define

$$\theta_n(f) = \mu(\theta_r(i_0 \cdots i_0 b_1 \cdots b_w i_0 \cdots i_0), \theta_{n-r}(c_1 \cdots c_s)). \quad (3.3.18)$$

(b) Now we assume that there exists some integers i and j such that $x_i = \varphi(t_j)$

In this case, there is a finite number of possibilities (depending of the fact that we consider the interval $[\varphi(t_j), x_i]$ or $[x_i, \varphi(t_j)]$ in the family $\{L_l\}_{l=0}^m$) to define the word f . It is not difficult to show, by using the naturality of the maps $a_r^\bullet \colon \Delta^\bullet \longrightarrow \mathcal{O}^\bullet$ and the fact that $\mu(\mu, id) = \mu(id, \mu)$ (we have this equality because $\mathcal{O}$ is a multiplicative operad), that all these possibilities lead to the same element $\theta_n((x, \widetilde{\varphi}), (a_1^\bullet, \cdots, a_n^\bullet))_k(t) \in \mathcal{O}(k)$. A good illustration of that is given by the second part of Example 3.3.12 below.

We can check that the maps $\theta_n((x, \widetilde{\varphi}), (a_1^\bullet, \cdots, a_n^\bullet))_k \colon \Delta^k \longrightarrow \mathcal{O}^k$ thus defined are continuous and commute with cofaces and codegeneracies. The continuity comes essentially from the fact that $\mu(\mu, id) = \mu(id, \mu)$. The continuity of the map $\theta_n \colon MS(n) \times (\mathrm{Tot}\mathcal{O}^\bullet)^n \longrightarrow \mathrm{Tot}\mathcal{O}^\bullet$ also comes from the same fact. $\square$

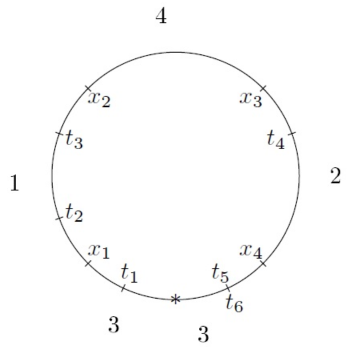


Figure a

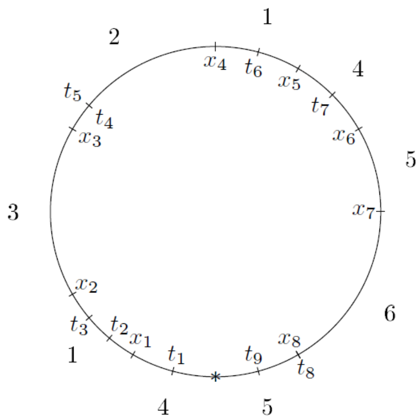


Figure b

Example 3.3.12. *(a) Let $n = 4$, $x \in \mathcal{I}_2(4)$ and $t = (t_1, \cdots, t_6) \in \Delta^6$ (see Figure a). Assume that $\widetilde{\varphi} = id_{S^1}$. Then $k = 6$, $p = 5$, $m = k + p - 1 = 10$ and the map $f\colon [10] \longrightarrow \{1, 2, 3, 4\}$ is defined by the word $f = 33111422333$. Now let us explicitly define $y_1 \in \Delta^{\left|f^{-1}(1)\right|} = \Delta^2$. First of all, we have $I_1 = [x_1, x_2]$. So the map g (see (3.3.12)) is just equal to g_{i_0} (notice that here $i_0 = 1$), and by the definition of this latter map (see (3.3.10)) we have $g_{i_0}(z) = z - (x_1 + 1)$ for each $z \in [x_1, x_2]$. On the other hand, the map α here is defined by $\alpha(z) = 4z + 3$ (see (3.3.13)). Therefore the image of each $z \in [x_1, x_2]$ under the composite $\widetilde{g} = \alpha \circ g_{i_0}$ gives $4z - 4x_1 - 1$. Hence,*

$$y_1 = (4t_2 - 4x_1 - 1, 4t_3 - 4x_1 - 1) \in \Delta^2$$

A similar computation gives $y_2 = 4t_4 - 4x_3 - 1 \in \Delta^1$. For y_3 we use the formula (3.3.14), and we obtain

$$y_3 = (4t_1 + 3, 4x_1 + 3, 4t_5 - 4x_4 + 4x_1 + 3, 4t_6 - 4x_4 + 4x_1 + 3) \in \Delta^4.$$

Now we can define $\theta_4(f)$.

$$\begin{aligned}
\theta_4(f) &= a_3^4(y_3) \circ_2 \theta_3(111244) \quad \textit{by (3.3.17)} \\
&= a_3^4(y_3) \circ_2 \mu(a_1^2(y_1), \theta_2(244)) \quad \textit{by (3.3.18)} \\
&= a_3^4(y_3) \circ_2 \mu(a_1^2(y_1), \mu(a_4^0(*), a_2^1(y_2))) \in \mathcal{O}(6) \quad \textit{by (3.3.15).}
\end{aligned}$$

(b) Let $n = 6$, $x \in \mathcal{I}_2(6)$ and $t = (t_1, \cdots, t_9) \in \Delta^9$ (see Figure b above). Assume that $\widetilde{\varphi} = id_{S^1}$. Then $k = 9$, $p = 9$ and $m = k + p - 1 = 17$. Since $t_8 = x_8$, it follows that there are two possibilities to define the map $f\colon [17] \longrightarrow \{1, 2, 3, 4, 5, 6\}$.
If $f = 441113222114456655$ then we have (by applying formulas (3.3.18), (3.3.17), (3.3.15) and (3.3.16))

$$\theta_6(f) = \mu(a_4^3(y_4) \circ_2 (a_1^4(y_1) \circ_3 \mu(a_3^0(*), a_2^2(y_2))), a_5^2(y_5) \circ_1 a_6^1(y_6)) \in \mathcal{O}(9).$$

Here $y_5 = (-1 \leq y_{51} \leq y_{52} \leq 1)$ is an element of Δ^2. Let us denote it by y_5^2.

If $f = 441113222114456555$ *then we have (again by formulas (3.3.18), (3.3.17), (3.3.15) and (3.3.16))*

$$\theta_6(f) = \mu(a_4^3(y_4) \circ_2 (a_1^4(y_1) \circ_3 \mu(a_3^0(*), a_2^2(y_2))), a_5^3(y_5) \circ_1 a_6^0(*)) \in \mathcal{O}(9).$$

Here $y_5 = (-1 \leq y_{50} \leq y_{51} \leq y_{52} \leq 1)$ *is an element of* Δ^3 *with* $y_{50} = y_{51}$. *Let us denote it by* y_5^3.

To check that these two possibilities lead to the same operation in $\mathcal{O}(9)$, *it suffices to check that*

$$a_5^3(y_5^3) \circ_1 a_6^0(*) = a_5^2(y_5^2) \circ_1 a_6^1(y_6).$$

Here we go

$$\begin{array}{rcl} a_5^3(y_5^3) \circ_1 a_6^0(*) & = & a_5^3(y_{50}, y_{51}, y_{52}) \circ_1 a_6^0(*) \\ & = & a_5^3(d^1(y_{51}, y_{52})) \circ_1 a_6^0(*) \ \ \textit{because} \ \ y_{50} = y_{51} \\ & = & (d^1(a_5^2(y_{51}, y_{52})) \circ_1 a_6^0(*) \ \ \textit{by the naturality of} \ \ a_5^\bullet \\ & = & (a_5^2(y_{51}, y_{52}) \circ_1 \mu) \circ_1 a_6^0(*) \ \ \textit{by the definition of} \ d^1 \\ & = & a_5^2(y_{51}, y_{52}) \circ_1 (\mu \circ_1 a_6^0(*)) \ \ \textit{by the associativity} \\ & = & a_5^2(y_{51}, y_{52}) \circ_1 (d^1(a_6^0(*))) \ \ \textit{by the definition of} \ d^1 \\ & = & a_5^2(y_{51}, y_{52}) \circ_1 a_6^1(d^1(*)) \ \ \textit{by the naturality of} \ \ a_6^\bullet \\ & = & a_5^2(y_5^2) \circ_1 a_6^1(y_6). \end{array}$$

Example 3.3.13. *In this example, we are in the case (a) of the proof of Theorem 3.3.11. Take* $t = (t_1, \cdots, t_{14}) \in \Delta^{14}$ *such that* $t_i \neq t_j$ *whenever* $i \neq j$, *and take* $f = 1112244221111333555511$. *The operation* $\theta_5(f) \in \mathcal{O}(14)$ *is then defined by*

$$\begin{array}{rcl} \theta_5(f) & = & (a_1^8(y_1) \circ_7 \theta_2(3335555)) \circ_3 \theta_2(224422) \ \ \textit{by (3.3.17)} \\ & = & (a_1^8(y_1) \circ_7 \mu(a_3^2(y_3), a_5^3(y_5))) \circ_3 (a_2^3(y_2) \circ_2 a_4^1(y_4)), \end{array}$$

where the last equality comes from (3.3.15) and (3.3.16).

3.3.2 The McClure-Smith operad $\mathcal{D}_2$ and its action on Tot$\mathcal{O}^\bullet$

Here we recall the construction of the operad $\mathcal{D}_2$, which was introduced by McClure and Smith in [32]. We also give the details of its action on Tot$\mathcal{O}^\bullet$. We will write Δ_+ for the category $\Delta \cup \{\emptyset\}$, and a covariant functor $X^\bullet \colon \Delta_+ \longrightarrow \mathrm{Top}$ will be called an *augmented cosimplicial space*. In all this section, the standard cosimplicial space $\Delta^\bullet$ will be viewed as an augmented cosimplicial space with $\Delta^\emptyset = \emptyset$.

Let us begin with the definition of the augmented cosimplicial space

$$\Xi_n^2(X^\bullet, \cdots, X^\bullet) \colon \Delta_+ \longrightarrow \mathrm{Top}$$

in which we have n copies of $X^\bullet$.

Let $\overline{n}$ as in the previous section. Define Q_n to be the category whose objects are pairs (T, f), where T is an object of the category Δ_+ and $f \colon T \longrightarrow \overline{n}$ is a morphism in sets. A morphism from (T, f) to (T', f') consists of a morphism $g \colon T \longrightarrow T'$ in Δ_+ such that $f = f'g$. Define also Q_n^2 to be the full subcategory of Q_n consisting of pairs (T, f) such that $\mathrm{cplx}(f) \leq 2$. Consider now the diagram

$$\begin{tikzcd} Q_n^2 \arrow[r, hook, "\rho_2"] \arrow[d, "\phi_n"'] & Q_n \arrow[r, "\psi_n"] & \mathrm{Top}^n \arrow[r, "\prod_n"] & \mathrm{Top} \\ \Delta_+ \end{tikzcd}$$

in which

- ρ_2 is the inclusion functor,
- ψ_n is defined to be $\psi_n(T, f) = (X^{f^{-1}(1)}, \cdots, X^{f^{-1}(n)})$,
- $\prod_n$ is the product functor and
- ϕ_n is the projection on the first component.

The covariant functor $\Xi_n^2(X^\bullet, \cdots, X^\bullet) \colon \Delta_+ \longrightarrow \mathrm{Top}$ is defined to be the left Kan extension of the composite $\prod_n \circ \psi_n \circ \rho_2$ along ϕ_n. By the definition of a left Kan extension, the functor $\Xi_n^2(X^\bullet, \cdots, X^\bullet)$ is explicitly defined as follows.

Let S be an object of the category Δ_+. We want to define the space $\Xi_n^2(X^\bullet, \cdots, X^\bullet)(S)$ associated to S. First we define the category A_{nS} whose objects are triples (h, T, f) where T is an object in Δ_+, $h \colon T \longrightarrow S$ is a morphism in Δ_+, and (T, f) is an object in Q_n^2. A morphism from (h, T, f) to (h', T', f') consists of a morphism $g \colon T \longrightarrow T'$ such that $h = h'g$ and $f = f'g$. We will sometimes write $S \xleftarrow{h} T \xrightarrow{f} \overline{n}$ for an object (h, T, f) of the category A_{nS}. Next we define the functor $p_n \colon A_{nS} \longrightarrow Q_n^2$ by $p_n(h, T, f) = (T, f)$, and we consider the A_{nS}-diagram

$$F_{nS} = \prod_n \circ \psi_n \circ \rho_2 \circ p_n \colon A_{nS} \longrightarrow \mathrm{Top}.$$

The space $\Xi_n^2(X^\bullet, \cdots, X^\bullet)(S)$ is then explicitly defined to be the colimit of F_{nS}. That is,

$$\Xi_n^2(X^\bullet, \cdots, X^\bullet)(S) = \underset{A_{nS}}{\mathrm{colim}} F_{nS}. \tag{3.3.19}$$

Notice that $\Xi_0^2(X^\bullet, \cdots, X^\bullet)(S)$ is the one point space since each Cartesian product $\prod_{i \in \overline{0}} f^{-1}(i)$ is a point (because $\overline{0} = \emptyset$). Notice also that $\Xi_1^2(X^\bullet) = X^\bullet$. On morphisms, the functor $\Xi_n^2(X^\bullet, \cdots, X^\bullet) \colon \Delta_+ \longrightarrow \mathrm{Top}$ is defined in the obvious way. We are now ready to define the operad $\mathcal{D}_2$.

Definition 3.3.14. *For $n \geq 0$ the space $\mathcal{D}_2(n)$ is defined to be*

$$\mathcal{D}_2(n) = \mathrm{Tot}(\Xi_n^2(\Delta^\bullet, \cdots, \Delta^\bullet)) = \mathrm{Nat}(\Delta^\bullet, \Xi_n^2(\Delta^\bullet, \cdots, \Delta^\bullet)).$$

McClure and Smith show [32, Section 9] that the collection $\{\mathcal{D}_2(n)\}_{n \geq 0}$ is a topological operad. They also show [32, Theorem 9.1 (a)] that this operad is actually weakly equivalent in the category of operads to the operad B_2 of little 2-disks.

Definition 3.3.15. *An augmented cosimplicial space $X^\bullet$ is a* Ξ^2-algebra *or is endowed with a* Ξ^2-structure *if there is a family*

$$\{\Theta_n \colon \Xi^2_n(X^\bullet, \cdots, X^\bullet) \longrightarrow X^\bullet\}_{n\geq 0}$$

of natural transformations such that
- $\Theta_1 \colon \Xi^2_1(X_\bullet) = X_\bullet \longrightarrow X_\bullet$ *is the identity map;*
- *for each choice of* $j_1, \cdots, j_n \geq 0$*, there is a natural transformation*

$$\Gamma \colon \Xi^2_n(\Xi^2_{j_1}, \cdots, \Xi^2_{j_n}) \longrightarrow \Xi^2_{j_1+\cdots+j_n}$$

making the following diagram commutative

$$\begin{tikzcd}
\Xi^2_n(\Xi^2_{j_1}(X_\bullet, \cdots, X_\bullet), \cdots, \Xi^2_{j_n}(X_\bullet, \cdots, X_\bullet)) \arrow[r, "\Gamma"] \arrow[d, "\Xi^2_n(\Theta_{j_1}, \cdots, \Theta_{j_n})"'] & \Xi^2_{j_1+\cdots+j_n}(X_\bullet, \cdots, X_\bullet) \arrow[d, "\Theta_{j_1+\cdots+j_n}"] \\
\Xi^2_n(X_\bullet, \cdots, X_\bullet) \arrow[r, "\Theta_n"] & X_\bullet;
\end{tikzcd}$$

- *for each permutation* $\sigma \in \Sigma_n$*,* $\Theta_n \circ \sigma_* = \Theta_n$*.*

Definition 3.3.16. *An augmented cosimplicial space $X^\bullet$ is said to be* reduced *if $X^\emptyset$ is the one point space.*

Proposition 3.3.17. [32, Proposition 10.3] *A sequence $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ of topological spaces is endowed with a structure of multiplicative operad if and only if the associated reduced augmented cosimplicial space $\mathcal{O}^\bullet$ is a Ξ^2-algebra.*

In [32, Theorem 9.1 (b)], McClure and Smith prove that the operad $\mathcal{D}_2$ acts on $\mathrm{Tot}\mathcal{O}^\bullet$, when the reduced augmented cosimplicial space $\mathcal{O}^\bullet$ is built from a multiplicative operad $\mathcal{O}$. We now recall this action. To do that, we will define (for each $n \geq 0$) a map

$$\beta_n \colon \mathcal{D}_2(n) \times (\mathrm{Tot}\mathcal{O}^\bullet)^n \longrightarrow \mathrm{Tot}\mathcal{O}^\bullet.$$

Let $\alpha = \{\alpha_k\}_{k\geq 0} \in \mathcal{D}_2(n)$, and let $(a_1^\bullet, \cdots, a_n^\bullet) \in (\mathrm{Tot}\mathcal{O}^\bullet)^n$. We want to define $\beta_n(\alpha, (a_1^\bullet, \cdots, a_n^\bullet)) \in \mathrm{Tot}\mathcal{O}^\bullet$. Form the diagram

$$\Delta^\bullet \xrightarrow{\alpha} \Xi^2_n(\Delta^\bullet, \cdots, \Delta^\bullet) \xrightarrow{\prod_{i=1}^n a_i^\bullet} \Xi^2_n(\mathcal{O}^\bullet, \cdots, \mathcal{O}^\bullet) \xrightarrow{\Theta_n} \mathcal{O}^\bullet$$

in which

- the natural transformation $\prod_{i=1}^n a_i^\bullet$ is induced by $a_1^\bullet, \cdots, a_n^\bullet$, and
- Θ_n is furnished by the Ξ^2-structure on $\mathcal{O}^\bullet$ (we have such a structure by Proposition 3.3.17).

The natural transformation $\beta_n(\alpha, (a_1^\bullet, \cdots, a_n^\bullet)) \colon \Delta^\bullet \longrightarrow \mathcal{O}^\bullet$ is then defined to be the composite

$$\beta_n(\alpha, (a_1^\bullet, \cdots, a_n^\bullet)) = \Theta_n \circ \prod_{i=1}^n a_i^\bullet \circ \alpha.$$

One can interpret a Ξ^2-structure in another way. McClure and Smith [32] show that the following definition is equivalent to Definition 3.3.15.

Definition 3.3.18. *An augmented cosimplicial space $X^\bullet$ is equipped with a* Ξ^2-structure *if for each map $f\colon T \longrightarrow \overline{n}$ (here T is a totally ordered set and $n \geq 0$) with complexity ≤ 2, there exists a map*

$$\langle f\rangle : X^{f^{-1}(1)} \times \cdots \times X^{f^{-1}(n)} \longrightarrow X^T$$

such that the collection of maps $\{\langle f\rangle\}$ is

- consistent *in the sense that for every commutative diagram*

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "h"'] & \overline{n} \\ T' \arrow[ur, "g"'] & \end{tikzcd}$$

  in which f and g have complexity ≤ 2, the following diagram commutes.

$$\begin{tikzcd} \prod_{i=1}^n X^{f^{-1}(i)} \arrow[r, "\langle f\rangle"] \arrow[d, "\prod (h_i)_*"'] & X^T \arrow[d, "h_*"] \\ \prod_{i=1}^n X^{g^{-1}(i)} \arrow[r, "\langle g\rangle"] & X^{T'} \end{tikzcd}$$

  Here h_i is the restriction of h to $f^{-1}(i)$.

- commutative *in the sense that for each f with complexity ≤ 2 and each $\sigma \in \Sigma_n$, the following diagram commutes.*

$$\begin{tikzcd} \prod_{i=1}^n X^{f^{-1}(i)} \arrow[r, "\langle f\rangle"] \arrow[d, "s"'] & X^T \arrow[d, "="] \\ \prod_{i=1}^n X^{f^{-1}(\sigma(i))} \arrow[r, "\langle \sigma^{-1}\circ f\rangle"] & X^T \end{tikzcd}$$

  Here s is the obvious permutation of the factors.

- associative *in the sense that for every choice of f and $g_1, \cdots, g_n$ with complexity ≤ 2, the following diagram commutes.*

$$\begin{tikzcd} \prod_{i=1}^n \prod_{p=1}^{j_i} X^{g_i^{-1}(p)} \arrow[r, "\prod\langle g_i\rangle"] \arrow[d, "="'] & \prod_{i=1}^n X^{f^{-1}(i)} \arrow[d, "\langle f\rangle"] \\ \prod_{q=1}^j X^{g^{-1}(q)} \arrow[r, "\langle g\rangle"] & X^T. \end{tikzcd}$$

  Here the maps $g_i\colon f^{-1}(i) \longrightarrow \overline{j_i}$ determine in an evident way a map $g\colon T \longrightarrow \overline{j}$, where $j = \sum_i^n j_i$.

- unital *in the sense that there is an element* $\zeta \in X^{\emptyset}$ *satisfying the equality*

$$\langle \lambda_i \rangle \circ f(x_1, \cdots, x_{i-1}, \zeta, x_i, \cdots, x_{n-1}) = \langle f \rangle (x_1, \cdots, x_{i-1}, x_i, \cdots, x_{n-1})$$

for all $f \colon T \longrightarrow \overline{n-1}$ *with complexity* ≤ 2, *all* $i \in \overline{n}$, *and all choices of* $x_1, \cdots, x_{n-1}$. *Here* $\lambda_i \colon \overline{n-1} \longrightarrow \overline{n}$ *is the order-preserving monomorphism whose image does not contains* i.

We are going to interpret (in the new language of the Ξ^2-structure on $\mathcal{O}^\bullet$) the action of $\mathcal{D}_2$ on $\mathrm{Tot}\mathcal{O}^\bullet$. We need this interpretation because it will be used in the proof of Theorem 3.3.19 below. Let α, $a_1^\bullet, \cdots, a_n^\bullet$ as before. We want to construct a natural transformation

$$\{\beta_n(\alpha, (a_1^\bullet, \cdots, a_n^\bullet))_k \colon \Delta^k \longrightarrow \mathcal{O}^k\}_{k \geq 0}.$$

So let $[k]$ be an object of the category Δ, and let $t \in \Delta^k$. By Definition 3.3.14, α_k is a map from Δ^k to $\Xi_n^2(\Delta^\bullet, \cdots, \Delta^\bullet)([k])$. Recalling that this latter space is the colimit of certain $A_{n[k]}$-diagram (see (3.3.19) above), there exists an object

$$[k] \xleftarrow{h} T \xrightarrow{f} \overline{n}$$

in the category $A_{n[k]}$ such that $\alpha_k(t)$ is the equivalence class of some $\widetilde{\alpha}_k(t) \in \Delta^{f^{-1}(1)} \times \cdots \times \Delta^{f^{-1}(n)}$. That is,

$$\alpha_k(t) = [\widetilde{\alpha}_k(t)].$$

Define $\beta_n(\alpha, (a_1^\bullet, \cdots, a_n^\bullet))_k(t)$ to be the image of $\widetilde{\alpha}_k(t)$ under the composite

$$\begin{tikzcd}
\Delta^{f^{-1}(1)} \times \cdots \times \Delta^{f^{-1}(n)} \arrow[r, "(a_1^{f^{-1}(1)}, \cdots, a_n^{f^{-1}(n)})"] \arrow[ddr] & \mathcal{O}^{f^{-1}(1)} \times \cdots \times \mathcal{O}^{f^{-1}(n)} \arrow[d, "\langle f \rangle"] \\
& \mathcal{O}^T \arrow[d, "h_*"] \\
& \mathcal{O}^k,
\end{tikzcd}$$

where h_* is the map induced by h, and $\langle f \rangle$ is given by the Ξ^2-structure of $\mathcal{O}^\bullet$, which is itself induced by the multiplicative structure of the operad $\mathcal{O}$ (we will recall the construction of $\langle f \rangle$ [32, Section 10] in the following lines). That is,

$$\beta_n(\alpha, (a_1^\bullet, \cdots, a_n^\bullet))_k(t) = h_* \circ \langle f \rangle \circ (a_1^{f^{-1}(1)}, \cdots, a_n^{f^{-1}(n)})(\widetilde{\alpha}_k(t)). \quad (3.3.20)$$

It is straightforward to check that the map $\beta_n(\alpha, (a_1^\bullet, \cdots, a_n^\bullet))_k \colon \Delta^k \longrightarrow \mathcal{O}^k$ is well defined. It is also straightforward to check that the collection of maps $\{\beta_n\}_{n \geq 0}$ defines an action of $\mathcal{D}_2$ on $\mathrm{Tot}\mathcal{O}^\bullet$.

We now recall the construction of $\langle f \rangle$. Let $\mu \in \mathcal{O}(2)$ as in the proof of Theorem 3.3.11, and let us denote the operad structure of $\mathcal{O}$ by

$$\gamma \colon \mathcal{O}(n) \times \mathcal{O}(i_1) \times \cdots \times \mathcal{O}(i_n) \longrightarrow \mathcal{O}(i_1 + \cdots + i_n).$$

- If $f\colon [r+s+1] \longrightarrow \overline{2}$ is defined by the word $f = \underbrace{1\cdots 1}_{r+1}\underbrace{2\cdots 2}_{s+1}$, then $\langle f\rangle : \mathcal{O}^r \times \mathcal{O}^s \longrightarrow \mathcal{O}^{r+s+1}$ is defined by the formula

$$\langle f\rangle (x,y) = \mu(x, d^0 y). \tag{3.3.21}$$

- If $f\colon [2n+i_1+\cdots+i_n] \longrightarrow \overline{n+1}$ is on the form

$$f = 1\underbrace{2\cdots 2}_{i_1+1}1\underbrace{3\cdots 3}_{i_2+1}1\cdots 1\underbrace{n+1\cdots n+1}_{i_n+1}1,$$

then $\langle f\rangle : \mathcal{O}^n \times \mathcal{O}^{i_1} \times \cdots \times \mathcal{O}^{i_n} \longrightarrow \mathcal{O}^{2n+i_1+\cdots i_n}$ is defined by the formula

$$\langle f\rangle (x, y_1, \cdots, y_n) = \gamma(x, d^0 d^{i_1+1} y_1, \cdots, d^0 d^{i_n+1} y_n). \tag{3.3.22}$$

- For a general $f\colon T \longrightarrow \overline{n}$ of complexity less than or equal to 2, the map $\langle f\rangle : \mathcal{O}^{f^{-1}(1)} \times \cdots \times \mathcal{O}^{f^{-1}(1)} \longrightarrow \mathcal{O}^T$ is defined (as a "combination" of formulas (3.3.21) and (3.3.22)) by induction on $\|T\| = |T|-1$. We refer the reader to [32, Section 10] for that induction.

3.3.3 The compatiblity between the MS and the $\mathcal{D}_2$ actions on Tot$\mathcal{O}^\bullet$

The goal of this section is to prove Theorem D announced in the introduction of this thesis.

Recalling the definition of the cacti operad MS from Section 3.3.1, we have the following theorem in which the second part is a more precise formulation of Theorem D.

Theorem 3.3.19. *(a) There exists an isomorphism $q\colon MS \xrightarrow{\cong} \mathcal{D}_2$.*

(b) Let $\mathcal{O}^\bullet$ be a cosimplicial space defined by a multiplicative operad $\mathcal{O}$. Then, for each $n \geq 0$, the square

$$\begin{tikzcd}
MS(n) \times (\mathrm{Tot}\mathcal{O}^\bullet)^n \arrow[r, "\theta_n"] \arrow[d, "q_n \times id^n"'] & \mathrm{Tot}\mathcal{O}^\bullet \arrow[d, "id"] \\
\mathcal{D}_2(n) \times (\mathrm{Tot}\mathcal{O}^\bullet)^n \arrow[r, "\beta_n"] & \mathrm{Tot}\mathcal{O}^\bullet
\end{tikzcd}$$

commutes.

Proof. **Proof of (a)**. This part was proved in [41] by P. Salvatore. We will still recall the explicit construction of $q\colon MS \longrightarrow \mathcal{D}_2$ since we need it to prove part (b). To see that q is an isomorphism of operads, we will refer the reader to [41, Proposition 8.2].

For each $n \geq 0$, we will construct an isomorphism $q_n\colon MS(n) \longrightarrow \mathcal{D}_2(n)$ in such a way that the collection $q = \{q_n\}_{n\geq 0}\colon MS \longrightarrow \mathcal{D}_2$ turns out to be a morphism of operads. So let $n \geq 0$ be an integer.
If $n = 0$ then $q_0\colon * = MS(0) \longrightarrow \mathcal{D}_2(0) = *$ is the unique map from the one

point space to itself.
If $n = 1$ then, by (3.3.4), we have

$$MS(1) = \{\varphi\colon S^1 \longrightarrow S^1 \text{ such that } \varphi \text{ is weakly monotone and } \varphi(*) = *\}.$$

We also have (by Definition 3.3.14)

$$\mathcal{D}_2(1) = \mathrm{Nat}(\Delta^\bullet, \Xi_1^2(\Delta^\bullet)) = \mathrm{Nat}(\Delta^\bullet, \Delta^\bullet) = \mathrm{Tot}\Delta^\bullet.$$

Let $\varphi \in MS(1)$, and let $t = (-1 \leq t_1 \leq \cdots \leq t_k \leq 1) \in \Delta^k$. Define $q_1(\varphi)\colon \Delta^k \longrightarrow \Delta^k$ by

$$q_1(\varphi)(t) = (-1 \leq \varphi(t_1) \leq \cdots \leq \varphi(t_k) \leq 1).$$

Now we assume that $n \geq 2$. Let $(x, \widetilde{\varphi}) \in MS(n) = \mathcal{I}_2(n) \times \mathrm{Mon}(I, \partial I)$. Set $x = (I_1, \cdots, I_n)$. Our goal is to construct

$$q_n(x, \widetilde{\varphi}) \in \mathcal{D}_2(n) = \mathrm{Nat}(\Delta^\bullet, \Xi_k^2(\Delta^\bullet, \cdots, \Delta^\bullet)).$$

Let $[k] \in \Delta$. We want to build a map

$$G_x\colon \Delta^k \longrightarrow \Xi_n^2(\Delta^\bullet, \cdots, \Delta^\bullet)([k]).$$

So let $t = (-1 \leq t_1 \leq \cdots \leq t_k \leq 1) \in \Delta^k$. Consider families $\{J_i\}_{i=0}^m$ and $\{L_l\}_{l=0}^m$ of closed intervals as defined in the beginning of the proof of Theorem 3.3.11. Let us take back the diagram (see (3.3.9))

$$[k] \xleftarrow{h} [m] \xrightarrow{f} \overline{n}\ .$$

Clearly we have $\mathrm{cplx}(f) \leq 2$ (this is because $\mathrm{cplx}(x) \leq 2$ by Definition 3.3.4), and h is a morphism in the category Δ_+. Hence, the triple $(h, [m], f)$ is an object in the category $A_{n[k]}$. Recalling that (by (3.3.19) above)

$$\Xi_n^2(\Delta^\bullet, \cdots, \Delta^\bullet)([k]) = \mathrm{colim}_{A_{n[k]}} F_{n[k]},$$

we are going now to built an explicit element $G_x([k])(t)$ of the space

$$F_{n[k]}(h, [m], f) = \prod_{i=1}^{n} \Delta^{|f^{-1}(i)|-1}.$$

Let $i \in \overline{n}$. As in the proof of Theorem 3.3.11, we construct an element $y_i \in \Delta^{|f^{-1}(i)|-1}$ (see (3.3.14) for the definition of y_i). We thus have an element

$$y = (y_1, \cdots, y_n) \in \prod_{i=1}^{n} \Delta^{|f^{-1}(i)|-1}.$$

Define now $G_x([k])(t) \in \mathrm{colim}_{A_{n[k]}} F_{n[k]}$ to be the equivalence class of y. That is,

$$G_x([k])(t) = [(y_1, \cdots, y_n)].$$

It is straightforward to check that the family $G_x = \{G_x([k])\}_{k\geq 0}$ is a natural transformation. The map $q_n\colon MS(n) \longrightarrow \mathcal{D}_2(n)$ is then defined by

$$q_n(x, \widetilde{\varphi}) = G_x.$$

It is also straightforward to check that the map $q = \{q_n\}_{n\geq 0}\colon MS \longrightarrow \mathcal{D}_2$ respects the operad structure.

Proof of (b). The result follows immediately when $n = 0$.
Let $n \geq 1$, $a_1^\bullet, \cdots, a_n^\bullet \in \mathrm{Tot}\mathcal{O}^\bullet$ and $(x, \widetilde{f}) \in MS(n)$. We want to show that

$$\beta_n(G_x, (a_1^\bullet, \cdots, a_n^\bullet)) = \theta_n((x, \widetilde{\varphi}), (a_1^\bullet, \cdots, a_n^\bullet)) \in \mathrm{Tot}\mathcal{O}^\bullet.$$

To do that, we will prove the following equality (for each $k \geq 0$)

$$\beta_n(G_x, (a_1^\bullet, \cdots, a_n^\bullet))_k = \theta_n((x, \widetilde{\varphi}), (a_1^\bullet, \cdots, a_n^\bullet))_k \colon \Delta^k \longrightarrow \mathcal{O}^k.$$

Let $[k] \in \Delta$. If $k = 0$ then the desired equality follows.
Now take $k \geq 1$, and let $t = (t_1, \cdots, t_k) \in \Delta^k$. We have

$$\begin{array}{rcl} \beta_n(G_x, (a_1^\bullet, \cdots, a_n^\bullet))_k(t) & = & h_* \circ \langle f \rangle \circ (a_1^{f^{-1}(1)}, \cdots, a_n^{f^{-1}(n)})(y) \text{ by (3.3.20)} \\ & = & h_* \circ \langle f \rangle \, (a_1^{f^{-1}(1)}(y_1), \cdots, a_n^{f^{-1}(n)}(y_n)). \end{array}$$

To end the proof of this part, it suffices to get the following equality

$$h_* \circ \langle f \rangle \, (a_1^{f^{-1}(1)}(y_1), \cdots, a_n^{f^{-1}(n)}(y_n)) = \theta_n(f), \tag{3.3.23}$$

when

$$f = \underbrace{1 \cdots 1}_{r+1}\underbrace{2 \cdots 2}_{s+1} \qquad \text{and} \qquad f = 1\underbrace{2 \cdots 2}_{i_1+1}1\underbrace{3 \cdots 3}_{i_2+1}1 \cdots 1\underbrace{n+1 \cdots n+1}_{i_n+1}1.$$

If $f = \underbrace{1 \cdots 1}_{r+1}\underbrace{2 \cdots 2}_{s+1}$ then the map h in the diagram

$$[r+s] \xleftarrow{\ h\ } [r+s+1] \xrightarrow{\ f\ } \overline{2}$$

is equal to the codegeneracy map s^{r+1}. Recalling that $id \in \mathcal{O}(1)$ is the identity operation, we first have

$$\begin{array}{rcl} \langle f \rangle \, (a_1^r(y_1), a_2^s(y_2)) & = & \mu(a_1^r(y_1), d^0 a_2^s(y_2)) \text{ by (3.3.21)} \\ & = & \mu(a_1^r(y_1), \mu(id, a_2^s(y_2))) \in \mathcal{O}^{r+s+1}. \end{array}$$

Next, recalling that $e \in \mathcal{O}(0)$ is the distinguish operation in arity 0, we have

$$\begin{array}{rcl} h_*(\langle f \rangle \, (a_1^r(y_1), a_2^s(y_2))) & = & s^{r+1}(\langle f \rangle \, (a_1^r(y_1), a_2^s(y_2))) \text{ since } h_* = s^{r+1} \\ & = & (\mu(a_1^r(y_1), \mu(id, a_2^s(y_2)))) \circ_{r+1} e \\ & = & \mu(a_1^r(y_1), a_2^s(y_2)) \\ & = & \theta_2(f) \in \mathcal{O}^{r+s} \text{ by (3.3.15)}, \end{array}$$

thus giving the equality (3.3.23).

Now we assume that f is on the form

$$f = 1\underbrace{2\cdots 2}_{i_1+1}1\underbrace{3\cdots 3}_{i_2+1}1\cdots 1\underbrace{n+1\cdots n+1}_{i_n+1}1.$$

We will prove the equality (3.3.23) when $n=3$ and

$$f = 1\underbrace{2\cdots 2}_{3+1}1\underbrace{3\cdots 3}_{5+1}1\underbrace{4\cdots 4}_{1+1}1$$

for instance (the proof being the same for general n and f).
By the formula (3.3.22), we have the operation

$$\langle f\rangle\,(a_1^3(y_1), a_2^3(y_2), a_3^5(y_3), a_4^1(y_4))$$

in $\mathcal{O}^{15}$, which is equal to

$$\gamma(a_1^3(y_1), d^0d^4a_2^3(y_2), d^0d^6a_3^5(y_3), d^0d^2a_4^1(y_4)).$$

Since f is on the above form, it follows that the map h in the diagram

$$[9] \xleftarrow{h} [15] \xrightarrow{f} \overline{4}$$

is defined by the word $h = 0012333456788899$. It is not difficult to see that $h = s^{15}\circ s^{13}\circ s^{12}\circ s^6\circ s^5\circ s^1$. Therefore

$$h_*(\langle f\rangle\,(a_1^3(y_1), a_2^3(y_2), a_3^5(y_3), a_4^1(y_4))) = \gamma(a_1^3(y_1), a_2^3(y_2), a_3^5(y_3), a_4^1(y_4)).$$

Moreover we have

$$\begin{aligned}\gamma(a_1^3(y_1), a_2^3(y_2), a_3^5(y_3), a_4^1(y_4)) &= ((a_1^3(y_1)\circ_3 a_4^1(y_4))\circ_2 a_3^5(y_3))\circ_1 a_2^3(y_2)\\ &= \theta_4(f)\in\mathcal{O}^9 \ \text{ by (3.3.17)},\end{aligned}$$

which completes the proof. □

3.4 Gerstenhaber algebra structure on $H_*(\mathbf{Tot}\mathcal{O}^\bullet)$

In this short section we first define an action of the circle S^1 on the totalization $\mathrm{Tot}\mathcal{O}^\bullet$, and we prove that it is compatible with the $\mathcal{D}_2$ action defined in the previous section. Next we recall a result of Sakai, which states that the Hochschild homology $HH(S_*\mathcal{O})$ is isomorphic as a Gerstenhaber algebra to the homology $H_*(\mathrm{hoTot}\mathcal{O}^\bullet)$.

Proposition 3.4.1. *The circle S^1 acts on $\mathrm{Tot}\mathcal{O}^\bullet$ in the sense that there exists a map*

$$\theta\colon S^1\times(\mathrm{Tot}\mathcal{O}^\bullet)^2\longrightarrow \mathrm{Tot}\mathcal{O}^\bullet. \tag{3.4.1}$$

Proof. Let $\tau \in S^1, (a_1^\bullet, a_2^\bullet) \in (\mathrm{Tot}\mathcal{O}^\bullet)^2$ and $t = (t_1, \cdots, t_k) \in \Delta^k$. We want to construct an operation $\theta(\tau, (a_1^\bullet, a_2^\bullet))_k(t) \in \mathcal{O}^k$. Let us set $(\phi, id)(\tau) = (x, id) \in MS(2)$, where the map $(\phi, id)\colon S^1 \longrightarrow MS(2)$ is given by (3.3.6). Racalling the definition of the operation $\theta_2((x, id), (a_1^\bullet, a_2^\bullet))_k(t)$ from formulas (3.3.15) and (3.3.16), we define

$$\theta(\tau, (a_1^\bullet, a_2^\bullet))_k(t) = \theta_2((x, id), (a_1^\bullet, a_2^\bullet))_k(t),$$

thus giving the desired result. □

The following corollary is an immediate consequence of Proposition 3.4.1.

Corollary 3.4.2. *The square*

$$\begin{tikzcd} S^1 \times (\mathrm{Tot}\mathcal{O}^\bullet)^2 \arrow[r, "\theta"] \arrow[d, "(\phi,id)\times id^2"'] & \mathrm{Tot}\mathcal{O}^\bullet \arrow[d, "id"'] \\ MS(2) \times (\mathrm{Tot}\mathcal{O}^\bullet)^2 \arrow[r, "\theta_2"] & \mathrm{Tot}\mathcal{O}^\bullet \end{tikzcd}$$

commutes.

In the following theorem, which says that the S^1 action is compatible with the $\mathcal{D}_2$ action on $\mathrm{Tot}\mathcal{O}^\bullet$, the map $q_2\colon MS(2) \longrightarrow \mathcal{D}_2(2)$ is the one constructed in the proof of Theorem 3.3.19.

Theorem 3.4.3. *Under the hypothesis of Theorem 3.3.11, the following square*

$$\begin{tikzcd} S^1 \times (\mathrm{Tot}\mathcal{O}^\bullet)^2 \arrow[r, "\theta"] \arrow[d, "(q_2\circ(\phi,id))\times id^2"'] & \mathrm{Tot}\mathcal{O}^\bullet \arrow[d, "id"'] \\ \mathcal{D}_2(2) \times (\mathrm{Tot}\mathcal{O}^\bullet)^2 \arrow[r, "\theta_2"] & \mathrm{Tot}\mathcal{O}^\bullet \end{tikzcd}$$

commutes.

It is well known that the action of an operad P, which is weakly equivalent to the little 2-disks operad B_2, on a space X induces a Gerstenhaber algebra structure on the homology $H_*(X)$. If the family $\{\psi_n\colon P(n) \times X^n \longrightarrow X\}_{n\geq 0}$ denotes the action of P on X, if x_0 and x_1 respectively denote the generators in degree 0 and in degree 1 of the homology $H_*(P(2)) \cong H_*(S^1)$ (we have this isomorphism because $P(2)$ is weakly equivalent to $B_2(2)$ which is itself weakly equvalent to the circle S^1), then the product and the bracket on $H_*(X)$ are defined by

$$a \times b = H_*(\psi_2)(x_0 \otimes a \otimes b) \qquad \text{and} \qquad \{a, b\} = H_*(\psi_2)(x_1 \otimes a \otimes b).$$

We come back now to the S^1 and $\mathcal{D}_2$ actions on $\mathrm{Tot}\mathcal{O}^\bullet$. In Proposition 3.4.1 we have defined an action of S^1 on $\mathrm{Tot}\mathcal{O}^\bullet$. From this action, Sakai deduces a formula (see [38, Theorem 4.5]) for the bracket on the homology $H_*(\mathrm{Tot}\mathcal{O}^\bullet)$. Notice that this bracket coincides with the one induced by the $\mathcal{D}_2$ action on

$\mathrm{Tot}\mathcal{O}^\bullet$ because of Theorem 3.4.3. In the case where the cosimplicial space $\mathcal{O}^\bullet$ is fibrant, he proves the following theorem.

($\mathbf{G}_1$) The Hochschild homology $HH(S_*\mathcal{O})$ is equipped with the Gerstenhaber algebra structure induced by (3.2.5) and (3.2.6).

($\mathbf{G}_2$) The homology $H_*(\mathrm{Tot}\mathcal{O}^\bullet)$ is equipped with the Gerstenhaber algebra structure induced by the $\mathcal{D}_2$ action on $\mathrm{Tot}\mathcal{O}^\bullet$.

Theorem 3.4.4. [38] *There is an isomorphism of Gerstenhaber algebras between $HH(S_*\mathcal{O})$ and $H_*(\mathrm{Tot}\mathcal{O}^\bullet)$ when $HH(S_*\mathcal{O})$ is equipped with the Gerstenhaber algebra structure described by* ($\mathbf{G}_1$)*, and $H_*(\mathrm{Tot}\mathcal{O}^\bullet)$ is equipped with the one described by* ($\mathbf{G}_2$)*.*

Proof. The proof follows immediately from [38, Theorem 4.6, Theorem 4.7, Proposition 4.8]. □

In the case where $\mathcal{O}^\bullet$ is not fibrant, he [38] proves Theorem 3.4.5 below, which is just the homotopy version of Theorem 3.4.4. Before stating it, recall that in [32, Section 15] McClure and Smith construct an operad $\widetilde{\mathcal{D}}_2$ weakly equivalent to the little 2-disks operad. It is defined in the similar way as $\mathcal{D}_2$, except that we must replace in Definition 3.3.14 $\Delta^\bullet$ by $\widetilde{\Delta}^\bullet$ (see the end of Section 1.2.2 for the definition of $\widetilde{\Delta}^\bullet$). That is,

$$\widetilde{\mathcal{D}}_2(n) = \mathrm{Tot}(\Xi^2_n(\widetilde{\Delta}^\bullet, \cdots, \widetilde{\Delta}^\bullet)). \tag{3.4.2}$$

They show [32, Theorem 15.3] that $\widetilde{\mathcal{D}}_2$ acts on the homotopy totalization $\mathrm{hoTot}\mathcal{O}^\bullet$, and thus inducing a Gerstenhaber algebra structure on the homology $H_*(\mathrm{hoTot}\mathcal{O}^\bullet)$.

($\mathbf{G}_3$) The homology $H_*(\mathrm{hoTot}\mathcal{O}^\bullet)$ is equipped with the Gerstenhaber algebra structure induced by the $\widetilde{\mathcal{D}}_2$ action on $\mathrm{hoTot}\mathcal{O}^\bullet$.

Theorem 3.4.5. [38, Theorem 2.3] *There is an isomorphism of Gerstenhaber algebras between $HH(S_*\mathcal{O})$ and $H_*(\mathrm{hoTot}\mathcal{O}^\bullet)$ when $HH(S_*\mathcal{O})$ is equipped with the Gerstenhaber algebra structure described by* ($\mathbf{G}_1$)*, and $H_*(\mathrm{hoTot}\mathcal{O}^\bullet)$ is equipped with the one described by* ($\mathbf{G}_3$)*.*

Remark 3.4.6. *Notice that Theorem 3.4.5 with $\mathcal{O} = \mathcal{K}_d$, the Kontsevich operad, has been announced earlier by Salvatore in [40, Proposition 22].*

CHAPTER 4

Gerstenhaber algebra structure on $H_*(\overline{\mathbf{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$

4.1 Introduction

The goal of this chapter is to prove Theorem A and Theorem B announced in the introduction of this thesis. Most of the results of this chapter appear in our paper [47]. We conclude the chapter with a few computations of the Gerstenhaber algebra of long knots. Here is a more detailed summary of this chapter.

In this chapter, we prove (see Theorem 4.1.1 below) that the Kontsevich operad $\mathcal{K}_d$ (which was recalled in Section 2.2.1 from Chapter 2) is $\mathbb{R}$-formal as a multiplicative operad, when $d \geq 3$. This result has two strong consequences: the first one (Corollary 4.1.3) says that Sinha's cosimplicial space $\mathcal{K}_d^{\bullet}$, which was also recalled at the beginning of Section 2.3, is $\mathbb{R}$-formal. This $\mathbb{R}$-formality gives a very short proof of Theorem 2.4.1, which states that the H_*BKSS associated to $\mathcal{K}_d^{\bullet}$ collapses at the E^2 page rationally. The second consequence (Theorem 4.1.5) and the most important gives the Gerstenhaber algebra structure on the homology $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$ of the space of long knots. More precisely, it says that the isomorphism between the E^2 page and the homology $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$ respects the Gerstenhaber algebra structure, when $d \geq 4$.

We have seen in Theorem 2.4.1 that the H_*BKSS computing the rational homology of the space of long knots collapses at the E^2 page only for $d > 3$. Notice that our approach in this chapter also does the work for $d = 3$ since we have the formality of the Sinha cosimplicial space for $d \geq 3$.

At the end of Section 2.4 we have made a comment about the zigzag (2.4.1). Here we prove that the second vertical morphism of that zigzag can be replaced by a morphism of operads on the form $\mathcal{A}s \longrightarrow \mathcal{O}$, and this gives us the following result, which is one of the main results in this thesis.

Theorem 4.1.1. *For $d \geq 3$, the operads $S_*(\mathcal{K}_d; \mathbb{R})$ and $H_*(\mathcal{K}_d; \mathbb{R})$ are weakly equivalent as multiplicative operads .*

For the meaning of "weakly equivalent as multiplicative operads", see Definition 4.2.2.

Remark 4.1.2. *In* [27] *it is only proved that $S_*(\mathcal{K}_d; \mathbb{R})$ and $H_*(\mathcal{K}_d; \mathbb{R})$ are weakly equivalent as "up to homotopy multiplicative operads"* (Definition 4.2.2)*, when $d \geq 3$. Notice that this result is not proved for $d = 2$ but only for $d \geq 3$ (see [27, Theorem 1.4]).*

An immediate consequence of Theorem 4.1.1 is the following formality result.

Corollary 4.1.3. *For $d \geq 3$ Sinha's cosimplicial space $\mathcal{K}_d^\bullet$ is formal over* $\mathbb{R}$.

Our method enables us also to determine the Gerstenhaber structure on the homology of the space of long knots.

We explain now with which Gerstenhaber structures we endow

$$H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d); \mathbb{R}) \qquad \text{and} \qquad HH(H_*\mathcal{K}_d; \mathbb{R}).$$

McClure and Smith construct in [32] two operads, $\mathcal{D}_2$ (this operad was recalled in Section 3.3.2 from Chapter 3) and $\widetilde{\mathcal{D}}_2$ (see equation (3.4.2) from Chapter 3), both weakly equivalent to the little 2-disks operad B_2. They show that if a cosimplicial space $\mathcal{O}^\bullet$ is built from a multiplicative operad $\mathcal{O}$, then $\mathcal{D}_2$ acts on the totalization $\mathrm{Tot}\mathcal{O}^\bullet$ (this action was detailed in Section 3.3.2), and $\widetilde{\mathcal{D}}_2$ acts on the homotopy totalization $\mathrm{hoTot}\mathcal{O}^\bullet$ [32, Theorem 15.3]. If in addition $\mathcal{O}$ is reduced (that is, both $\mathcal{O}(0)$ and $\mathcal{O}(1)$ are weakly contractible), then the homotopy totalization of $\mathcal{O}^\bullet$ is weakly homotopy equivalent to the double loop space of a certain explicit space of maps of operads (Dwyer-Hess [11] and Turchin [52] prove this result by using different approaches). Notice that neither Dwyer-Hess nor Turchin actually prove that their delooping is the delooping with respect to the McClure-Smith $\widetilde{\mathcal{D}}_2$ action.

Let us come back now to the particular case of Kontsevich's operad $\mathcal{K}_d$, which is reduced by Remark 2.2.4. Since it is multiplicative by Proposition 2.2.8, it follows that the operad $\widetilde{\mathcal{D}}_2$ acts on $\mathrm{hoTot}\mathcal{K}_d^\bullet \simeq \overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$ (recall that this weak equivalence was reviewed in Section 2.3). We also have a B_2 geometric action (constructed by Budney in [8]) on (framed) long knots. One question arises: are these two actions equivalent? This question is still open to my knowledge. But one thing is certain: each of these actions induces a Gerstenhaber algebra structure on $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d); \mathbb{R})$, and apparently it has never been checked whether these structures coincide. We now specify which one we choose.

Consider the following three facts.

(A) The H_*BKSS associated to $\mathcal{K}_d^\bullet$ collapses (Theorem 2.4.1) at the E^2 page, and converges to the homology $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$ ([44, Theorem 7.2]). Associated to a multiplicative operad $\mathcal{B}_*(\bullet)$ in chain complexes is its Hochschild homology $HH(\mathcal{B}_*(\bullet))$, defined first by Gerstenhaber and Voronov in [14]. It is endowed with a natural Gerstenhaber algebra structure (see [14] or [40, Section 4] for more details about this natural structure).

(B) The E^2 page, which is isomorphic to $HH(H_*\mathcal{K}_d;\mathbb{R})$, is equipped with the natural Gerstenhaber algebra structure. This structure is in fact the one induced by formulas (3.2.5) and (3.2.6) from Chapter 3

(C) The homology $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R},\mathbb{R}^d))$ is equipped with the Gerstenhaber algebra structure induced by the action of $\widetilde{\mathcal{D}}_2$ on $\mathrm{hoTot}\mathcal{K}_d^\bullet \simeq \overline{\mathrm{Emb}}(\mathbb{R},\mathbb{R}^d)$.

The fact **(A)** implies that there is an isomorphism

$$\psi\colon E^2 \xrightarrow{\cong} H_*(\overline{\mathrm{Emb}}_c(\mathbb{R},\mathbb{R}^d))$$

of vector spaces. From **(A)**, **(B)** and **(C)**, we would like to directly deduce that the isomorphism ψ also respects the Gerstenhaber algebra structure. But the Example 1.5.3 from Chapter 1, as many other examples, prevents us from affirming that.

We can then ask the following question.

Question 4.1.4. *Does the isomorphism ψ respect the Gerstenhaber algebra structure?*

Our method enables us to answer this question.

Theorem 4.1.5. *For $d \geq 4$, there exists an isomorphism of Gerstenhaber algebras between the homology of space of long knots modulo immersions and the Hochschild homology $HH(H_*\mathcal{K}_d)$ associated to $H_*(\mathcal{K}_d;\mathbb{R})$ when the homology $H_*(\overline{\mathrm{Emb}}(\mathbb{R},\mathbb{R}^d);\mathbb{R})$ is equipped with the Gerstenhaber algebra structure described by* **(C)***, and $HH(H_*\mathcal{K}_d;\mathbb{R})$ is equipped with the one described by* **(B)***. That is,*

$$H_*(\overline{\mathrm{Emb}}(\mathbb{R},\mathbb{R}^d);\mathbb{R}) \cong HH(H_*\mathcal{K}_d;\mathbb{R}). \tag{4.1.1}$$

Remark 4.1.6. *For $d \geq 4$, it is proved in* [26] *that*

$$HH(H_*\mathcal{K}_d;\mathbb{Q}) \qquad \textit{and} \qquad H_*(\overline{\mathrm{Emb}}(\mathbb{R},\mathbb{R}^d);\mathbb{Q})$$

are isomorphic as vector spaces but not as Gerstenhaber *algebras.*

Remark 4.1.7. *When a version of this chapter was ready, Syunji Moriya put in arXiv a paper [34] in which equivalent results are independently discovered.*

Outline of the chapter

- In Section 4.2 we prove Lemma 4.2.3, which is crucial for the rest of the chapter.
- In Section 4.3 we apply Lemma 4.2.3 to the specific zig-zag between $S_*(\mathcal{K}_d;\mathbb{R})$ and its homology operad $H_*(\mathcal{K}_d;\mathbb{R})$, and we obtain Theorem 4.1.1. This result implies immediately that Sinha's cosimplicial space is formal over $\mathbb{R}$ (Corollary 4.1.3). Using now this formality, we give a very short proof of the collapse of the Vassiliev spectral sequence over rationals (Theorem 2.4.1). We end the section with the proof of Theorem 4.1.5, which is essentially a consequence of Theorem 4.1.1 and Theorem 3.4.5.

- In Section 4.4 we use our Theorem 4.1.5 to compute some homology classes of the space of long knots that are not directly induced by chord diagrams. We also give a little table for d odd. Using that table we show (Proposition 4.4.2) that the homology $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R},\mathbb{R}^d))$ is not free as a Gerstenhaber algebra, when $d \geq 4$ is odd.

4.2 Equivalences of multiplicative operads

The goal of this section is to state and prove the crucial Lemma 4.2.3. This lemma will be used in the next section to prove Theorem 4.1.1, which is one of the main results in this thesis. The category $\mathcal{C}$ here is as in Section 1.4 from Chapter 1, that is, $\mathcal{C}$ is a symmetric monoidal model category that is cofibrantly generated. Recall that by $\mathcal{C}^{\mathbb{N}}$ we denote the category of sequences in $\mathcal{C}$, and by $\mathcal{O}p_{ns}$ we denote the category of nonsymmetric operads in $\mathcal{C}$. Recall also that the associative operad $\mathcal{A}s$ in $\mathcal{C}$ was introduced in Example 1.4.11 from Chapter 1, where we have reviewed the homotopy theory for operads.

Let us begin with some necessary definitions.

Definition 4.2.1. *An* up-to-homotopy multiplicative operad *consists of a triple $(\mathcal{O},\mathcal{A},\eta)$ in which $\mathcal{O}$ is a nonsymmetric operad in $\mathcal{C}$, $\mathcal{A}$ is an operad weakly equivalent to the associative operad $\mathcal{A}s$, and $\eta : \mathcal{A} \longrightarrow \mathcal{O}$ is a morphism of nonsymmetric operads.*

Notice that by Definition 1.4.13 from Chapter 1, every multiplicative operad is an up-to-homotopy multiplicative operad. Therefore the category of multiplicative operads is a full subcategory of the category of up-to-homotopy multiplicative operads. In the latter category, a morphism from $(\mathcal{O},\mathcal{A},\eta)$ to $(\mathcal{O}',\mathcal{A}',\eta')$ consists of morphisms $g : \mathcal{O} \longrightarrow \mathcal{O}'$ and $f : \mathcal{A} \longrightarrow \mathcal{A}'$ such that $g\eta = \eta' f$.

Definition 4.2.2. *Two multiplicative operads $\mathcal{M}$ and $\mathcal{M}'$ are said to be* weakly equivalent as multiplicative operads *(respectively* weakly equivalent as up-to-homotopy multiplicative operads*) if there is a zig-zag*

$$\mathcal{M} \xleftarrow{\sim} \mathcal{O}_1 \xrightarrow{\sim} \cdots \xleftarrow{\sim} \mathcal{O}_p \xrightarrow{\sim} \mathcal{M}'$$

in the category of multiplicative operads (respectively in the category of up-to-homotopy multiplicative operads).

We are ready to state and prove our crucial lemma.

Lemma 4.2.3. *In the category $\mathcal{O}p_{ns}$ of nonsymmetric operads in $\mathcal{C}$, consider the following commutative diagram*

$$\begin{tikzcd}
\mathcal{M}_1 & \mathcal{O} \arrow[l, "\sim", "f_1"'] \arrow[r, "\sim"', "f_2"] & \mathcal{M}_2 \\
\mathcal{A}s \arrow[u, "\alpha_1"'] & \mathcal{A} \arrow[l, "\sim", "\sigma"'] \arrow[r, "\sim", "\sigma"'] \arrow[u, "\eta"'] & \mathcal{A}s. \arrow[u, "\alpha_2"']
\end{tikzcd}$$

Assume that $\mathcal{A}$ is cofibrant as an object of $\mathcal{C}^{\mathbb{N}}$. Then the operads $\mathcal{M}_1$ and $\mathcal{M}_2$ are weakly equivalent as multiplicative operads.

Proof. We begin by the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}s \arrow[r, "\alpha_1"] & \mathcal{M}_1 \\
\mathcal{A} \arrow[u, "\sigma", "\sim"'] \arrow[r, "\eta"] \arrow[d, "\sigma"', "\sim"] & \mathcal{O} \arrow[u, "f_1", "\sim"'] \arrow[d, "f_2"', "\sim"] \\
\mathcal{A}s \arrow[r, "\alpha_2"] & \mathcal{M}_2
\end{tikzcd}$$

Since the object $\mathcal{A}$ is cofibrant in the category $\mathcal{C}^{\mathbb{N}}$, by applying the factorization axiom to the morphism $\eta : \mathcal{A} \longrightarrow \mathcal{O}$, we obtain the diagram

$$\begin{tikzcd}
\mathcal{A}s \arrow[rr, "\alpha_1"] & & \mathcal{M}_1 \\
\mathcal{A} \arrow[u, "\sigma", "\sim"'] \arrow[rr, "\eta"] \arrow[dr, tail, "\eta_1"] \arrow[dd, "\sigma"', "\sim"] & & \mathcal{O} \arrow[u, "f_1", "\sim"'] \arrow[dd, "f_2"', "\sim"] \\
& Y \arrow[ur, two heads, "\eta_2", "\sim"'] & \\
\mathcal{A}s \arrow[rr, "\alpha_2"] & & \mathcal{M}_2
\end{tikzcd}$$

By taking the pushout of the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, tail, "\eta_1"] \arrow[d, "\sigma"', "\sim"] & Y \\
\mathcal{A}s &
\end{tikzcd}$$

we obtain

$$\begin{tikzcd}
\mathcal{A}s \arrow[rr, "\alpha_1"] & & \mathcal{M}_1 \\
\mathcal{A} \arrow[u, "\sigma", "\sim"'] \arrow[rr, "\eta"] \arrow[dr, tail, "\eta_1"] \arrow[ddd, "\sigma"', "\sim"] & & \mathcal{O} \arrow[u, "f_1", "\sim"'] \arrow[ddd, "f_2"', "\sim"] \\
& Y \arrow[ur, two heads, "\eta_2", "\sim"'] \arrow[d, "g"', "\sim"] & \\
& \widetilde{\mathcal{O}} \arrow[dr, dotted] & \\
\mathcal{A}s \arrow[ur, "\tilde{\eta}"] \arrow[rr, "\alpha_2"] & & \mathcal{M}_2 .
\end{tikzcd}$$

Since the operad $\mathcal{A}s$ is cofibrant in $\mathcal{C}^{\mathbb{N}}$(see Remark 1.4.12 from Chapter 1) and $\mathcal{A}$ is also cofibrant in $\mathcal{C}^{\mathbb{N}}$ by hypothesis, and since the morphism $\sigma : \mathcal{A} \longrightarrow \mathcal{A}s$ is

a weak-equivalence and the morphism $\eta_1 : \mathcal{A} \longrightarrow Y$ is a cofibration, it follows by Proposition 1.4.6 (again from Chapter 1) that the morphism $g : Y \longrightarrow \widetilde{\mathcal{O}}$ is a weak-equivalence.
Consider now the following pushout diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, tail, "\eta_1"] \arrow[d, "\sigma"', "\sim"] & Y \arrow[d, "g", "\sim"'] \arrow[ddr, bend left, "f_2\eta_2", "\sim"'] & \\
\mathcal{A}s \arrow[r, "\tilde{\eta}"] \arrow[drr, bend right, "\alpha_2"'] & \widetilde{\mathcal{O}} \arrow[dr, dotted] & \\
 & & \mathcal{M}_2.
\end{tikzcd}$$

The universal property of the pushout and the two-out-of-three axiom M2 allow us to obtain a weak-equivalence

$$\tilde{f}_2 : \widetilde{\mathcal{O}} \xrightarrow{\sim} \mathcal{M}_2.$$

Similarly, by considering the pushout diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, tail, "\eta_1"] \arrow[d, "\sigma"', "\sim"] & Y \arrow[d, "g", "\sim"'] \arrow[ddr, bend left, "f_1\eta_2", "\sim"'] & \\
\mathcal{A}s \arrow[r, "\tilde{\eta}"] \arrow[drr, bend right, "\alpha_1"'] & \widetilde{\mathcal{O}} \arrow[dr, dotted] & \\
 & & \mathcal{M}_1.
\end{tikzcd}$$

we deduce the existence of a weak-equivalence

$$\tilde{f}_1 : \widetilde{\mathcal{O}} \xrightarrow{\sim} \mathcal{M}_1.$$

Finally, we obtain the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}_1 & \widetilde{\mathcal{O}} \arrow[l, "\tilde{f}_1"', "\sim"] \arrow[r, "\tilde{f}_2", "\sim"'] & \mathcal{M}_2 \\
\mathcal{A}s \arrow[u, "\alpha_1"'] & \mathcal{A}s \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\widetilde{\eta}"'] & \mathcal{A}s. \arrow[u, "\alpha_2"']
\end{tikzcd}$$

□

4.3 Formality of the Kontsevich operad as a multiplicative operad

The goal of this section is to prove Theorem 4.1.1, Corollary 4.1.3 and Theorem 4.1.5 announced in the introduction of this chapter. We will also give a very short proof of Theorem 2.4.1. The ground field in this section is $\mathbb{R}$. Recall that the Kontsevich operad $\mathcal{K}_d$ was defined in Section 2.2.1 from Chapter 2.

4.3.1 Proof of Theorem 4.1.1

To prove Theorem 4.1.1, we need to detail the zig-zag of quasi-isomorphisms connecting the singular chains $S_*(\mathcal{K}_d;\mathbb{R})$ to the homology $H_*(\mathcal{K}_d;\mathbb{R})$ of the Kontsevich operad $\mathcal{K}_d$. Since the operad $C_*(\mathcal{K}_d;\mathbb{R})$ of semi-algebraic chains on $\mathcal{K}_d$ appears in that zig-zag, it will be more convenient to first recall the notion of *semi-algebraic chains* (a good reference for semi-algebraic chains is [20]). A *semi-algebraic set* is a subset of $\mathbb{R}^p$ that is obtained by finite unions, finite intersection, and complements of subsets defined by polynomial equations and inequalities. A *semi-algebraic map* is a continuous map between semi-algebraic sets whose graph is a semi-algebraic set. A *semi-algebraic manifold of dimension k* is semi-algebraic set such that each point has a semi-algebraic neighborhood semi-algebraically homeomorphic to $\mathbb{R}^k$ or to $\mathbb{R}_+ \times \mathbb{R}^{k-1}$. Let SemiAlg denote the category of semi-algebraic sets. It turns out to be a symmetric monoidal category with the Cartesian product as the tensor product, and the one point space as the unit. There is an obvious functor from SemiAlg to topological spaces: the forgetful functor

$$U\colon \mathrm{SemiAlg} \longrightarrow \mathrm{Top},$$

which is of course a symmetric monoidal functor.

Example 4.3.1. *The Kontsevich operad $\mathcal{K}_d$ and the Fulton-MacPherson operad $\mathcal{F}_d$ (see Section 2.2.2 for the definition of $\mathcal{F}_d$) are operads in* SemiAlg.

In [20, Section 3], one constructs a functor

$$C_*(-;\mathbb{R})\colon \mathrm{SemiAlg} \longrightarrow \mathrm{Ch}_{\mathbb{R}},$$

called the *functor of semi-algebraic chains*. For a semi-algebraic set X, a typical element of $C_k(X)$ is represented by a semi-algebraic map $\sigma\colon M \longrightarrow X$ from a semi-algebraic compact oriented manifold M of dimension k to X. So the functor C_* is much like the classical functor of singular chains

$$S_*(-;\mathbb{R})\colon \mathrm{Top} \longrightarrow \mathrm{Ch}_{\mathbb{R}}.$$

Notice that the functor C_* is a symmetric monoidal functor because of [20, Proposition 3.8] (this implies that $C_*(\mathcal{K}_d)$ and $C_*(\mathcal{F}_d)$ are operads in semi-algebraic chains). Notice also that the composite $S_* \circ U\colon \mathrm{Semialg} \longrightarrow \mathrm{Ch}_{\mathbb{R}}$ is symmetric monoidal. We thus have two symmetric monoidal functors $C_*, S_* \circ U\colon \mathrm{Semialg} \longrightarrow \mathrm{Ch}_{\mathbb{R}}$, which are weakly equivalent by Proposition 7.2 in [20]. In the proof of that proposition, one constructs a zig-zag

$$S_*(X) \xleftarrow{\sim} S_*^{PA}(X) \xrightarrow{\sim} C_*(X)$$

for every semi-algebraic set X. Here the chain complex $S_*^{PA}(X)$ of semi-algebraic singular chains is defined as the normalized chain complex associated to the simplicial set $S_\bullet^{PA}(X)$ with

$$S_k^{PA}(X) = \{\sigma\colon \Delta^k \longrightarrow X \mid \sigma \text{ is a semi-algebraic map}\}.$$

Note that the functor $S_*^{PA} \colon \mathrm{SemiAlg} \longrightarrow \mathrm{Ch}_{\mathbb{R}}$ is also symmetric monoidal.

Let us come back now to the formality of the operad of little d-disks, and consider the Theorem 2.4.3 (which says in particular that the little d-disks operad is relatively formal over the real numbers when $d \geq 3$) from Chapter 2. The authors of [27] prove that theorem by explicitly constructing a diagram

$$\begin{tikzcd}
S_*(\mathcal{K}_d) & S_*^{PA}(\mathcal{K}_d) \arrow[l, "\sim"'] \arrow[r, "\sim"] & C_*(\mathcal{K}_d) & C_*(\mathcal{F}_d) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathcal{D}_d^\vee & H_*(\mathcal{K}_d) \arrow[l, "\sim"'] \\
S_*(\mathcal{K}_1) \arrow[u] & S_*^{PA}(\mathcal{K}_1) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & C_*(\mathcal{K}_1) \arrow[u] & C_*(\mathcal{F}_1) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & \mathcal{D}_1^\vee \arrow[u] & H_*(\mathcal{K}_1). \arrow[l, "\sim"'] \arrow[u]
\end{tikzcd}$$

In that diagram, the quasi-isomorphism $C_*(\mathcal{F}_d) \xrightarrow{\sim} \mathcal{D}_d^\vee$ is built in [27, Chapter 9], and the quasi-isomorphism $H_*(\mathcal{K}_d) \xrightarrow{\sim} \mathcal{D}_d^\vee$ is built in [27, Chapter 8] (we precise that the morphism $C_*(\mathcal{F}_1) \xrightarrow{\sim} \mathcal{D}_1^\vee$ is specially defined in [27, Chapter 10]). Again in the same diagram, the quasi-isomorphism $C_*(\mathcal{K}_d) \xleftarrow{\sim} C_*(\mathcal{F}_d)$ is obtained by applying the functor C_* to the diagram (2.2.7) from Chapter 2.

Before starting the proof of Theorem 4.1.1, we recall that the unit $\mathbf{1}_*$ in the symmetric monoidal category $(\mathrm{Ch}_{\mathbb{R}}^{\mathbb{N}}, \otimes, \mathbf{1}_*)$ of chain complexes is defined by

$$\mathbf{1}_* = \begin{cases} \mathbb{R} & \text{if} \quad * = 0 \\ 0 & \text{if} \quad * \geq 1. \end{cases}$$

This implies that the associative operad in chain complexes is $\mathcal{A}s = \{\mathbf{1}_*\}_{n \geq 0}$.

Proof of Theorem 4.1.1. Since

- the operads $S_*(\mathcal{K}_d)$, $S_*^{PA}(\mathcal{K}_d)$, $H_*(\mathcal{K}_d)$ are all multiplicative (by Proposition 1.4.15 and Proposition 2.2.8), and $S_*(\mathcal{K}_1)$, $S_*^{PA}(\mathcal{K}_1)$, $H_*(\mathcal{K}_1)$ are all the associative operad because $S_*(-), S_*^{PA}(-), H_*(-)$ are symmetric monoidal functors and $\mathcal{K}_1$ is the associative operad (Remark 2.2.7 from Chapter 2 claims that $\mathcal{K}_1 = \mathcal{A}s$);
- the operad $\mathcal{D}_d^\vee$ is multiplicative by Proposition 2.2.22,

we can consider the following subdiagram of the big diagram above

$$\begin{tikzcd}
C_*(\mathcal{K}_d) & C_*(\mathcal{F}_d) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathcal{D}_d^\vee \\
C_*(\mathcal{K}_1) \arrow[u] & C_*(\mathcal{F}_1) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & \mathcal{D}_1^\vee. \arrow[u]
\end{tikzcd} \tag{4.3.1}$$

In the lower row of (4.3.1),

- $C_*(\mathcal{F}_1)$ is not the associative operad because, for instance, $\mathcal{F}_1(3)$ is homeomorphic to the unit interval I of $\mathbb{R}$ and the chain complex $C_*(I)$ is not the unit $\mathbf{1}_*$,

- $C_*(\mathcal{K}_1)$ is the associative operad because the functor C_* is symmetric monoidal and $\mathcal{K}_1$ is the associative operad,

- $\mathcal{D}_1^\vee$ is the associative operad too because of equation (2.2.10) from Chapter 2,

- the two morphisms from $C_*(\mathcal{F}_1)$ to $\mathcal{A}s$ are the same because they agree in degree 0,

- the objects $C_*(\mathcal{K}_1)$ and $C_*(\mathcal{F}_1)$ are cofibrant in the model category $\mathrm{Ch}_{\mathbb{R}}^{\mathbb{N}}$ of nonsymmetric sequences in $\mathrm{Ch}_{\mathbb{R}}$. This comes from Example 1.3.2 and the fact that the model structure on $\mathrm{Ch}_{\mathbb{R}}^{\mathbb{N}}$ is levelwise (that is, weak equivalences, fibrations and cofibrations are all levelwise).

Applying now Lemma 4.2.3 with the diagram (4.3.1), we get the desired result. □

4.3.2 Proof of Corollary 4.1.3 and Theorem 2.4.1

Recalling the Definition 2.4.4 from Chapter 2 of the formality of a cosimplicial space, we have the following proof.

Proof of Corollary 4.1.3 and Theorem 2.4.1. By Theorem 4.1.1 the operads $S_*(\mathcal{K}_d)$ and $H_*(\mathcal{K}_d)$ are weakly equivalent as multiplicative operads. Therefore the associated cosimplicial objects $(S_*(\mathcal{K}_d))^\bullet$ and $(H_*(\mathcal{K}_d))^\bullet$ are weakly equivalent in the category of cosimplicial chain complexes over $\mathbb{R}$, hence $S_*(\mathcal{K}_d^\bullet)$ is formal over $\mathbb{R}$. Now the collapsing of the Bousfield Kan spectral sequence comes from the fact that in the E^2 page we can replace the column $S_*(\mathcal{K}_d^p)$ by the homology $H_*(\mathcal{K}_d^p)$, hence the vertical differential vanishes and the spectral sequence collapses over $\mathbb{R}$ (see [26, Proposition 3.2]). Notice that it also collapses over $\mathbb{Q}$ because of the following isomorphism (recalling the notation $E^r(-)$ introduced in the last section of Chapter 1)

$$\{E^r(S_*(\mathcal{K}_d^\bullet;\mathbb{R}))\}_{r\geq 0} \cong \{E^r(S_*(\mathcal{K}_d^\bullet;\mathbb{Q}))\otimes_{\mathbb{Q}}\mathbb{R}\}_{r\geq 0}.$$

□

4.3.3 Gerstenhaber algebra structure on the homology of the space of long knots

The goal here is to prove Theorem 4.1.5 announced in the introduction.

In [33] McClure and Smith construct an E_2 chain operad $\mathcal{T}_2$ (which is the algebraic version of the operad $\mathcal{D}_2$ defined in Section 3.3.2) that acts on the Hochschild complex $CH(V)$ (well defined in Section 3.2), when V is an operad with multiplication in chain complexes (recall that an E_2 *chain operad* is a chain operad weakly equivalent to the normalized singular chain of the little 2-disks operad B_2). This action induces a Gerstenhaber algebra structure on the Hochschild homology $HH(V)$. It is very important to note that this structure coincides with the natural one (the one induced by formulas (3.2.5) and (3.2.6) from Chapter 3) because $\mathcal{T}_2$ is a solution of Deligne's conjecture. Let $\mathcal{T}_2$-algebras denote the category of chain complexes equipped with an action of the operad $\mathcal{T}_2$. Let $\mathcal{O}p_*(\mathrm{Ch}_\mathbb{R})$ denote the category of multiplicative nonsymmetric operads in chain complexes over real numbers.

Lemma 4.3.2. [33] *There exists a functor*

$$CH \colon \mathcal{O}p_*(\mathrm{Ch}_\mathbb{R}) \longrightarrow \mathcal{T}_2\text{-algebras}$$

that preserves weak equivalences.

Recalling ($\mathbf{G}_3$) from the end of Chapter 3, we have the following proposition.

Proposition 4.3.3. *Let $\mathcal{O}$ be a multiplicative operad, and $\mathcal{O}^\bullet$ be the associated cosimplicial space. Assume that operads $S_*(\mathcal{O})$ and $H_*(\mathcal{O})$ are weakly equivalent as multiplicative operads. Then there exists an isomorphism of Gerstenhaber algebras between the Hochschild homology $HH(H_*(\mathcal{O})$ and the homology $H_*(\mathrm{hoTot}\mathcal{O}^\bullet)$ when $HH(H_*(\mathcal{O}))$ is equipped with the natural Gerstenhaber algebra structure and $H_*(\mathrm{hoTot}\mathcal{O}^\bullet)$ is equipped with the one described by* ($\mathbf{G}_3$).

Proof. First, in the category of multiplicative operads in chain complexes, we have by hypothesis a zig-zag

$$S_*(\mathcal{O}) \xleftarrow{\sim} \cdots \xrightarrow{\sim} H_*(\mathcal{O}). \tag{4.3.2}$$

Next, by applying the normalized Hochschild complex functor CH to (4.3.2), we obtain a zig-zag

$$CH(S_*(\mathcal{O}) \xleftarrow{\sim} \cdots \xrightarrow{\sim} CH(H_*(\mathcal{O})) \tag{4.3.3}$$

in the category of $\mathcal{T}_2$-algebras by Lemma 4.3.2. Therefore the homology of (4.3.3) gives

$$HH(S_*(\mathcal{O})) \xleftarrow{\cong} \cdots \xrightarrow{\cong} HH(H_*(\mathcal{O}))\ , \tag{4.3.4}$$

which respects the Gerstenhaber algebra structure induced by the $H_*(\mathcal{T}_2)$ action. Finally, the desired result follows from (4.3.4) and Theorem 3.4.5. □

We are now ready to prove Theorem 4.1.5.

Proof of Theorem 4.1.5. It suffices to apply Proposition 4.3.3 with $\mathcal{O} = \mathcal{K}_d$. □

4.4 Some homology classes in the space of long knots

It is well known that one can compute a number of homology classes of the space $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$ of long knots by the mean of *chord diagrams* (see Cattaneo-Cotta-Ramusino-Longoni [9]). Such homology classes live in degrees multiple of $d-3$ (here we assume $d \geq 4$). In this section we will compute, using our Theorem 4.1.5, some non trivial homology classes that are not directly induced by chord diagrams. As in the previous section, the ground field here is $\mathbb{R}$. Note that we can also make computations over $\mathbb{Q}$ because of the rational collapsing.

We have seen in Theorem 4.1.5 that the H_*BKSS computing the homology $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d); \mathbb{R})$ of the space of long knots (modulo immersions) collapses at the E^2 page as a gerstenhaber algebra. Therefore, to compute the produit and the bracket on $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d); \mathbb{R})$, it suffices to do it at the E^2 page level, which is isomorphic to the Hochschild homology $HH(H_*\mathcal{K}_d)$. Using now the weak equivalence

$$\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d) \simeq \mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d) \times \Omega^2 S^{d-1} \tag{4.4.1}$$

from equation (2.1.2), one can deduce the homology $H_*(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d))$ since the homology $H_*(\Omega^2 S^{d-1})$ is well known. Recall that it is the free graded commutative algebra generated by u and $\{u, u\}$ (u is the diagram of Figure 2 below, and the bracket here is the one induced by the natural action of the little 2-disks operad on $\Omega^2 S^{d-1}$) if d is odd, and by u otherwise.
Recall that the above E^2 page was extensively studied by Turchin in [54]. He makes computations at that page by using the fact that the homology $H_*(\mathcal{K}_d(-p))$ can be replaced by a complex of chord diagrams with p vertices. For instance, when $p = -2$, the homology $H_*(\mathcal{K}_d(-p)) \cong H_*(S^{d-1})$ (since $\mathcal{K}_d(2) \simeq \mathrm{Conf}(2, \mathbb{R}^d) \simeq S^{d-1}$) is generated by the following diagrams

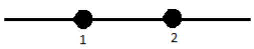


Figure 1

Figure 2

The first diagram is in bidegree $(-2, 0)$ and the second one in bidegree $(-2, d-1)$. In fact, one defines the bidegree of a diagram Γ with k chords and j vertices by the formula

$$(-j, k(d-1)).$$

The correponding degree of Γ in the homology of the space of long knots is then $k(d-1) - j$. For instance the diagram of Figure 3 below is of bidegree $(-4, 2(d-1))$, and it lives in degree $2(d-3)$ in the homology $H_*(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d))$.

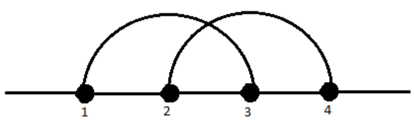


Figure 3

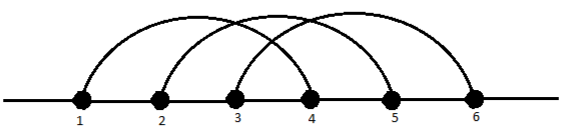


Figure 4

Let $v \in E^1_{-4,2(d-1)}$ denote the diagram of Figure 3. Note that $u \in E^1_{-2,d-1}$, and recall that the E^1 page is equipped with the horizontal differential of bidegree $(-1, 0)$ obtained by taking the alternate sum of cofaces after normalizing (see the end of Section 1.5.1 for the notion of normalization of a cosimplicial chain complex). Since there is nothing to kill u, it suvives at the E^2 page and represents the generator of $H_{d-3}(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$. Hence,

$$H_{d-3}(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d); \mathbb{R}) \cong \mathbb{R}.$$

Notice that this generator comes from the second factor of (4.4.1). This implies that $H_{d-3}(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)) = 0$. By a similar reasoning, we have

$$H_{2d-6}(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d); \mathbb{R}) \cong \mathbb{R}^2.$$

Here the generators are u^2 and v, and therefore we have an isomorphism (since u^2 lives in $H_*(\Omega^2 S^{d-1})$)

$$H_{2d-6}(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)) \cong \mathbb{R}.$$

More generally, there is at least one diagram that belongs to $E^1_{-2k,k(d-1)}$ (for instance, the diagram of Figure 4 when $k = 3$), survives at the E^2 page and which is a generator of $H_{k(d-3)}(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d))$. We thus obtain generators of $H_*(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d))$ in degrees $k(d-3), k \geq 0$. In [9], Cattaneo-Cotta-Ramusino-Longoni obtain these generators using another approach. Their idea consists to associate from a chord diagram with k chords a long immersion with k transversal self-intrsections points, and then "solve" them. Since the resolution of each of these points is parametrized by the $d-3$ dimensional sphere S^{d-3}, it follows that there is a map $(S^{d-3})^k \longrightarrow \mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$. Taking the image of the fundamental class of $(S^{d-3})^k$ under the induced map in homology, one obtains the above generators.

All the generators we have seen until now live in degrees multiple of $d-3$, and are induced directly by chord diagrams. By using the bracket $\{-, -\}$ of $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$, we can produce new homology classes. Indeed, by [54, Equation (2.9.21)], there are two generators $\{u, v\}$ and $\{u, u^2\}$ in $E^1_{-5,3(d-1)}$ (or only one generator depending of the parity of d) that survive in the E^2 page and represent the generators of $H_{3d-8}(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$. More precisely, we have an isomorphism

$$H_{3d-8}(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d); \mathbb{R}) \cong \begin{cases} \mathbb{R}^2 & \text{if } d \text{ odd} \\ \mathbb{R} & \text{if } d \text{ even.} \end{cases}$$

This is because by the biderevative of the bracket with respect to the product, and by the graded commutativity of the product (see Section 3.2), we have

$$\{u, u^2\} = \begin{cases} 2u\{u, u\} & \text{if } d \text{ odd} \\ 0 & \text{if } d \text{ even} \end{cases}$$

As before, only the generator $\{u, u^2\}$ comes from the second factor of (4.4.1) when d is odd. Hence, we have

$$H_{3d-8}(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d); \mathbb{R}) \cong \mathbb{R}.$$

By using our Theorem 4.1.5, it is possible to compute more homology classes in high degrees (see the explicit computations that Turchin makes at the E^2 page in [54] and [55]).

Let us end the section with a remark, a table and a proposition. In [40, Theorem 1] Salvatore proves that the space $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$ of long knots is a double loop space for $d > 3$. Notice that the double loop space structure is not induced directly by an operad as hoped in [44], but by a fibration argument of a diagram of cosimplicial spaces. On the other hand, the space $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$ is a double loop space because of the weak equivalence $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d) \simeq \mathrm{hoTot}\mathcal{K}_d^\bullet$ (see Theorem 2.3.3) and Theorem 15.3 of [32]. Consider now the projection

$$p\colon \overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d) \longrightarrow \mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d).$$

It is natural to ask whether p preserves the double loop space structure. This question was studied by Salvatore [40, Theorem 2], who obtained a negative answer in odd dimension. His argument lies on the fact that in homology, p_* sends u to zero by the dimensional reason, and sends the bracket $\{u, v\}$ to the generator of $H_{3d-8}(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d); \mathbb{R})$. Notice that in even dimension the morphism $E^2(p)$ at the E^2 pages level respects the Gerstenhaber algebra structure. Hence, the obstruction argument does not work in that dimension.

Remark 4.4.1. *Salvatore did not need our Theorem 4.1.5 to conclude that the bracket $\{u, v\}$ is not equal to 0 in the homology $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$ of the space of long knots. Moreover he did not need the collapse of the Vassiliev spectral sequence (see Theorem 2.4.1). His argument is the following. Consider a filtration of $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$ that gives the E^∞ page, which contains the bracket $\{u, v\}$ since we are in low degree and there is no differential to kill him. This bracket is not zero (because of Turchin computations [54]) in the quotient associated to the filtration. Therefore, by a simple argument of spectral sequences, it is not zero in the homology $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$.*

The following conjectural table is made for d odd and concerns the homology

$$H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)) \cong H_*(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)) \otimes H_*(\Omega^2 S^{d-1})$$

of the space of long knots modulo immersions. It follows from Turchin computations (see Table 4 and Table 6 in [55]) on the E^2 page and from our Theorem 4.1.5. In that table, the abbreviation N.G means number of generators, and the abbreviation N.P.G means number of primitive generators (for us, a *primitive generator* is generator that is not decomposable as a product of generators of degree greater than or equal to 1).

Degree	Generators	N.G	N.P.G
0	1	1	0
$d-3$	u	1	1
$2d-6$	u^2, v	2	1
$2d-5$	$\{u,u\}$	1	1
$3d-9$	u^3, uv, w	3	1
$3d-8$	$u\{u,u\}, \{u,v\}$	2	1
$4d-12$	$u^4, v^2, uw, u^2v, x, x'$	6	2
$4d-11$	$u^2\{u,u\}, v\{u,u\}, u\{u,v\}, \{v,v\}, \{u,w\}$	5	2
$5d-15$	$u^5, uv^2, u^2w, u^3v, vw, ux, ux', y, y', y''$	10	3
$5d-14$	$u\{u,w\}, u\{v,v\}, u^2\{u,v\}, u^3\{u,u\}, v\{u,v\}, w\{u,u\}, uv\{u,u\}, \{v,w\}, \{u,x\}, \{u,x'\}$	10	3

In that table we can easily see that the homology classes $\{u, v\}$ (in degree $3d-8$), $\{v, v\}$ and $\{u, w\}$ (in degree $4d-11$), $\{v, w\}$, $\{u, x\}$ and $\{u, x'\}$ (in degree $5d-14$) live in the homology $H_*(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d))$ of the space $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^d)$ of long knots, and do not come directly from chord diagrams. These are primitive generators (see Turchin's Table 6 in [55]).

In the Table above, we easily see that the bracket $\{u, \{u, v\}\}$ is equal to zero because it lives in degree $4d-10$, and there is nothing there. This leads us to the following result.

Proposition 4.4.2. *For odd $d \geq 4$, the homology $H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$ is not free as a Gerstenhaber algebra.*

Remark 4.4.3. *For $d = 3$, Budney [8] shows that the Gerstenhaber algebra* [1] *$H_*(\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d))$ is free.*

[1] Here the Gerstenhaber algebra structure is induced by the action of the little 2-disks operad. The question to know whether the Budney action and the McClure-Smith action are equivalent is still open

CHAPTER 5

Rational homology of spaces of long links

5.1 Introduction

This chapter proves Theorem E, Theorem F, Theorem G and Theorem I announced in the introduction of this thesis. All these theorems, except Theorem I, appear in our preprint [49], which determines the rational homology of the space of long links. Here is a more detailed summary of this chapter.

Let us start with the definition of a *long link*. As in Chapter 2 and Chapter 4, the integer $d \geq 3$ still denotes the dimension of the ambient space. Roughly speaking, if one thinks a long knot (see Definition 2.1.1) as a strand or a string, the definition of a *long link* is a generalization of Definition 2.1.1 to multiple strands. More precisely, let $m \geq 1$ be an integer (m represents the number of strands), and let $\{a_1, \cdots, a_m\}$ be a family of real numbers inside $I = [-1, 1]$ defined by

$$a_i = \frac{2i - m - 1}{m + 1}. \tag{5.1.1}$$

This family defines in fact a partition of I into intervals with equal lengths. Define $\overline{m} = \{1, \cdots, m\}$, and fix a family $\{\epsilon_i \colon \mathbb{R} \hookrightarrow \mathbb{R}^d\}$ of linear embeddings defined by

$$\epsilon_i(t) = (0, \cdots, 0, a_i, -t),\ 1 \leq i \leq m.$$

Definition 5.1.1. *A* long link of m strands *in* $\mathbb{R}^d$ *is a smooth embedding* $f \colon \mathbb{R} \times \overline{m} \hookrightarrow \mathbb{R}^d$ *of* m *copies of* $\mathbb{R}$ *inside* $\mathbb{R}^d$ *satisfying the boundary conditions*

$$\begin{cases} f(I \times \overline{m}) \subseteq \mathbb{R}^{d-1} \times I \\ f(t, i) = \epsilon_i(t), 1 \leq i \leq m \quad \textit{if} \quad |t| \geq 1. \end{cases} \tag{5.1.2}$$

Notice that this definition coincides with that of a long knot, when $m = 1$ (one can observe that the equation (5.1.1) above gives $a_1 = 0$ in that case). Let us define now the space we study in this chapter.

Definition 5.1.2. *The* space of long links of m strands *is the collection*

$$\{f\colon \mathbb{R}\times\overline{m}\hookrightarrow\mathbb{R}^d \text{ such that } f \text{ is a long link of } m \text{ strands}\}$$

*endowed with the weak $\mathcal{C}^\infty$-topology (for that topology, see **(WT)** from the introduction of Chapter 2).*

We denote this space by $\mathrm{Emb}_c(\coprod_1^m\mathbb{R},\mathbb{R}^d)$ or simply by $\mathcal{L}_m^d$. Define the space $\mathrm{Imm}_c(\coprod_1^m\mathbb{R},\mathbb{R}^d)$ of long immersions of m strands analogously. It is clear that there is an inclusion $\mathrm{Emb}_c(\coprod_1^m\mathbb{R},\mathbb{R}^d)\hookrightarrow\mathrm{Imm}_c(\coprod_1^m\mathbb{R},\mathbb{R}^d)$, and its homotopy fiber is called the *space of long links modulo immersions*. Let us denote this latter space, which is the object of our study in this chapter, by $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^m\mathbb{R},\mathbb{R}^d)$ or simply by $\overline{\mathcal{L}}_m^d$.

In this chapter we provide a complete understanding of the rational homology of the space $\overline{\mathcal{L}}_m^d$, when $d>5$. First, we construct explicitly a cosimplicial chain complex $L_*^\bullet$ whose totalization is quasi-isomorphic to the singular chain complex of the space of long links. Next we show (using the fact that the Bousfield-Kan spectral sequence associated to $L_*^\bullet$ collapses at the E^2 page) that the homology Bousfield-Kan spectral sequence associated to the Munson-Volić cosimplicial model for the space of long links collapses at the E^2 page rationally, and this solves a conjecture of Munson-Volić. Our method enables us also to obtain the collapsing at the E^2 page of the spectral sequence computing the rational homology of the space $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^m\mathbb{R}^n,\mathbb{R}^d)$, which is the high dimensional analogues of spaces of long links. The last result of the chapter states that the radius of convergence of the Poincaré series for the space $\overline{\mathcal{L}}_m^d$ and for the pair $(\overline{\mathcal{L}}_m^d,(\overline{\mathcal{L}}_1^d)^{\times m})$ tends to 0 when m goes to the infinity.

Now we are going to state properly the main results of the chapter. As in Chapter 2, let $\mathrm{Conf}(k,\mathbb{R}^d)$ denote the space of configuration of k points in $\mathbb{R}^d$. We will construct an explicit cosimplicial chain complex $L_*^\bullet$, where

$$L_*^p=H_*(\mathrm{Conf}(mp,\mathbb{R}^d);\mathbb{Q}).$$

As in the previous chapters, let B_d denote the little d-disks operad (recall that this operad was introduced in Example 1.4.10), and let s^{-p} be the suspension functor of degree $-p$ (it was defined in equation (3.2.1) from Chapter 3). Define the totalization $\mathrm{Tot}L_*^\bullet$ by

$$\mathrm{Tot}L_*^\bullet=\bigoplus_{p\geq 0}(s^{-p}L_*^p),$$

the differential being the alternating sum of cofaces. We will see in Section 5.3 that the homology $H_*(\mathrm{Tot}L_*^\bullet)$ can be interpreted by the $\vee_{i=1}^m S^1$-homology of $H_*(B_d;\mathbb{Q})$.

Here is the first result of the chapter, which says that the cosimplicial chain complex $L_*^\bullet$ gives a cosimplicial model for the singular chain complex of the space of long links.

Theorem 5.1.3. *For $d>5$, the totalization of $L_*^\bullet$ is weakly equivalent to the singular chain complex of the space of long links of m strands in $\mathbb{R}^d$. That is,*

$$\mathrm{Tot}L_*^\bullet\simeq S_*(\overline{\mathcal{L}}_m^d)\otimes\mathbb{Q}.$$

The following corollary is an immediate consequence of Theorem 5.1.3.

Corollary 5.1.4. *For $d > 5$, the rational homology of the space of long links of m strands is isomorphic to the $\vee_{i=1}^{m} S^1$-homology of $H_*(B_d;\mathbb{Q})$. That is,*

$$H_*(\overline{\mathcal{L}}_m^d;\mathbb{Q}) \cong HH^{\vee_{i=1}^{m} S^1}(H_*(B_d;\mathbb{Q})).$$

For the meaning of the *X-homology of something*, see Definition 5.3.5.

Let us state now the second and the most important result of the chapter, which is also one of the main results of this thesis. Actually, this result solves (for $d > 5$) a conjecture of Munson-Volić that we state now. In [35], B. Munson and I. Volić built a cosimplicial space (we denote it by $\mathcal{L}_m^{d\bullet}$) that gives a cosimplicial model for the space $\overline{\mathcal{L}}_m^d$ of long links of m strands in $\mathbb{R}^d$, when $d \geq 4$. They also define two spectral sequences that converge respectively to the homotopy and cohomology of the space $\overline{\mathcal{L}}_m^d$ of long links modulo immersions. In this chapter, we look at the homology Bousfield-Kan spectral sequence associated to $\mathcal{L}_m^{d\bullet}$, which converges to the homology $H_*(\overline{\mathcal{L}}_m^d)$ by Theorem 5.1.3.

Conjecture 5.1.5. [Munson-Volić] *This spectral sequence collapses at the E^2 page rationally for $d \geq 4$.*

Theorem 5.1.6. *For $d > 5$, the homology Bousfield-Kan spectral sequence associated to the Munson-Volić cosimplicial model $\mathcal{L}_m^{d\bullet}$ for the space of long links of m strands in $\mathbb{R}^d$ collapses at the E^2 page rationally.*

Remark 5.1.7. *Our method enables us also to determine the rational homology of the high dimensional analogues of spaces $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^{m} \mathbb{R}^n, \mathbb{R}^d)$ of long links modulo immersions. More precisely, as in the case of long links, we construct an explicit cosimplicial chain complex $L_*^{n\bullet}$ and we prove (in the similar way as Theorem E) that it gives a cosimplicial model for the singular chain complex of $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^{m} \mathbb{R}^n, \mathbb{R}^d)$. We thus obtain Theorem 5.5.2, Corollary 5.5.3 and Proposition 5.5.4.*

The case $m = 1$ (this case corresponds to the space $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$ of long knots) was studied in the previous chapters. Here is a summary of obtained results.
First, Sinha constructs in [44] a cosimplicial model $\mathcal{K}_d^\bullet$ of the space of long knots $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$, when $d \geq 4$. Next, Lambrechts, Turchin and Volić prove [26] that the H_*BKSS associated to $\mathcal{K}_d^\bullet$ collapses at the E^2 page rationally, when $d \geq 4$. Few years later, we prove (see Corollary 4.1.3) that the collapsing result still holds for $d \geq 3$, and thus simplify the proof of the main result of [26].
Again for $m = 1$, the spectral sequence $\{E^r(L_*^\bullet)\}_{r\geq 0}$ computing the rational homology of $\overline{\mathcal{L}}_1^d$ is isomorphic from the E^1 page to the H_*BKSS associated to $\mathcal{K}_d^\bullet$, therefore Theorem 5.1.6 is a generalization of the above results.
Recall other interesting results obtained in the study of the space of long knots. We have discovered in Theorem 4.1.1 the *multiplicative formality* (for $d \geq 3$) of the Kontsevich operad $\mathcal{K}_d(\bullet)$. This result gave us (for $d \geq 3$) the formality of Sinha's cosimplicial space $\mathcal{K}_d^\bullet$, and Theorem 4.1.5 (this theorem furnishes a

complete understanding of the rational homology of the space of long knots as a Gerstenhaber algebra).

We end this introduction with the last result of the chapter, which concerns the *Poincaré series* for the space of long links. In [24] Komawila and Lambrechts study the E^2 page of the cohomology Bousfield-Kan spectral sequence associated to the Munson-Volić cosimplicial space. They show that the coefficients of the associated Euler series have an exponential growth of rate $m^{\frac{1}{d-1}} > 1$. Using now our collapsing Theorem 5.1.6, we deduce the following result.

Theorem 5.1.8. *For $d > 5$ the radius of convergence of the Poincaré series for the space of long links (modulo immersions) $\overline{\mathcal{L}}_m^d$ is less than or equal to $(\frac{1}{m})^{\frac{1}{d-1}}$. Therefore the Betti numbers of $\overline{\mathcal{L}}_m^d$ have an exponential growth.*

An immediate consequence of Theorem 5.1.8 is the following corollary.

Corollary 5.1.9. *For $d > 5$ the previous radius of convergence tends to 0 as m goes to ∞.*

When $m = 1$ the upper bound of Theorem 5.1.8 is equal to 1, and the following theorem, due to Turchin, gives a better upper bound in that case.

Theorem 5.1.10. [53] *For $d \geq 4$ the radius of convergence of the Poincaré series for the space of long knots (modulo immersions) is less than or equal to $(\frac{1}{\sqrt{2}})^{\frac{1}{d-1}}$.*

Since the space of m copies of long knots is a retraction up to homotopy of the space of long links, Theorem 5.1.10 implies that the radius of convergence of the Poincaré series for $\overline{\mathcal{L}}_m^d$ is less than or equal to $(\frac{1}{\sqrt{2}})^{\frac{1}{d-1}}$. Our Corollary 5.1.9 gives a better upper bound for m large.

Outline of the chapter

- In Section 5.2 we first define a manifold M, and prove (Proposition 5.2.2) that the study of the space of long links (modulo immersions) is reduced to the study of the space $\overline{\mathrm{Emb}}_c(M, \mathbb{R}^d)$ of compactly supported embeddings of M into $\mathbb{R}^d$. Next we recall the notion of *infinitesimal bimodules* with a lot of examples. We also recall some results (related to the Taylor tower associated to $\overline{\mathrm{Emb}}_c(\mathbb{R}^n, \mathbb{R}^d)$ and $S_*\overline{\mathrm{Emb}}_c(\mathbb{R}^n, \mathbb{R}^d)$) obtained by Arone and Turchin in [2]. These results say that the kth approximation of the Taylor tower can be expressed in terms of morphisms of *infinitesimal bimodules.* Finally we show that similar results (Proposition 5.2.24, Proposition 5.2.25, Proposition 5.2.27 and Proposition 5.2.28) hold for the space $\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d)$, where N is the complement of a compact subset of $\mathbb{R}^q$.

- In Section 5.3 we construct an explicit cosimplicial chain complex $L_*^\bullet$ that gives a cosimplicial model for $S_*(\overline{\mathrm{Emb}}_c(M, \mathbb{R}^d); \mathbb{Q})$ (this is Theorem 5.1.3). To prove Theorem 5.1.3, we will use all the results obtained in Section 5.2.3 , and also Proposition 5.3.6, Lemma 5.3.2 and Theorem 5.3.7.

- In Section 5.4 we first prove (Lemma 5.4.1) that the E^1 pages of spectral sequences $\{E^r(L^\bullet_*)\}_{r\geq 0}$ and $\{E^r(S_*(\mathcal{L}^{d\bullet}_m;\mathbb{Q}))\}_{r\geq 0}$ are isomorphic. Using now the fact that the spectral sequence $\{E^r(L^\bullet_*)\}_{r\geq 0}$ collapses at the E^2 page (Lemma 5.4.2) and Theorem 5.1.3, we deduce Theorem 5.1.6.
- In Section 5.5 we show (using our approach) that the spectral sequence computing the rational homology of the high dimensional analogues of spaces of long links collapses at the E^2 page.
- In Section 5.6 we show that the radius of convergence of the Poincaré series for the space of long links modulo m copies of the space of long knots tends to 0 when m goes to the infinity. To get this we use our Theorem 5.1.6 and a theorem of Komawila-Lambrechts [24].

5.2 A compactly supported version of Goodwillie-Weiss embedding calculus for the space of long links

We introduce this section with a proposition, which allows us to reduce the study of the space $\overline{\mathcal{L}}^d_m$ to the study of the space $\overline{\mathrm{Emb}}_c(M,\mathbb{R}^d)$ of compactly supported embeddings of a manifold M into $\mathbb{R}^d$. Before stating this proposition, we properly define our manifold M, which is roughly speaking the complement in $\mathbb{R}^2$ of a slightly thickening of m copies of the interval $I=[-1,1]$. The ground field in this section is $\mathbb{Q}$.

Let I be the interval $I=[-1,1]$, and let $m\geq 1$ be an integer. Consider the familly $\{a_0,a_1,\cdots,a_m\}\subseteq I$ defined by (5.1.1) above. Let $0<\epsilon<\frac{2}{m+1}$ be a fixed real number. For $0\leq i\leq m$, define

$$K_{in}=I^n\times[a_i,a_i+\epsilon],\qquad \text{and}\qquad K_n=\cup_{i=o}^m K_{in}.$$

Definition 5.2.1. *We define M to be the complement of K_1 in $\mathbb{R}^2$. That is,*

$$M=\mathbb{R}^2\backslash K_1. \tag{5.2.1}$$

To study the rational homology of high dimensional analogues of spaces of long links, we will consider the manifold M_n defined by

$$M_n=\mathbb{R}^{n+1}\backslash K_n. \tag{5.2.2}$$

Proposition 5.2.2. *For $d\geq 3$, the space of long links modulo immersions in $\mathbb{R}^d$ is weakly equivalent to the space of smooth compactly supported embeddings of M in $\mathbb{R}^d$,*

$$\overline{\mathcal{L}}^d_m\simeq\overline{\mathrm{Emb}}_c(M,\mathbb{R}^d).$$

Proof. Since $\overline{\mathcal{L}}^d_m(\mathbb{R})$ is the space of compactly supported embeddings of m copies of $\mathbb{R}$ in $\mathbb{R}^d$, it follows that it is weakly equivalent to the space of embeddings (with endpoints and tangent vectors at those endpoints fixed on oppposite faces of the cube) modulo immersions of m copies of the interval I in the cube I^d,

$$\overline{\mathcal{L}}^d_m\simeq\overline{\mathrm{Emb}}_c(\coprod_1^m I,I^d).$$

Moreover, for each $0 \leq i \leq m-1$, the subspace of I^2, which is between K_i and K_{i+1}, is weakly equivalent to I. This ends the proof. $\square$

Remark 5.2.3. *The Proposition 5.2.2 is easily generalized to the high dimensional analogue $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^m \mathbb{R}^n, \mathbb{R}^d)$ of the space of long links. That is, we have the following weak equivalence*

$$\overline{\mathrm{Emb}}_c(\coprod_{i=1}^m \mathbb{R}^n, \mathbb{R}^d) \simeq \overline{\mathrm{Emb}}_c(M_n, \mathbb{R}^d).$$

Here and in the rest of this chapter, $S_*(-)$ is the normalized singular chain functor. The advantage to work with the space $\overline{\mathrm{Emb}}_c(M_n, \mathbb{R}^d)$ instead of the space $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^m \mathbb{R}^n, \mathbb{R}^d)$ is the fact that we can directly use the same techniques (which was developed by Arone and Turchin [2] in the study of the space $\overline{\mathrm{Emb}}_c(\mathbb{R}^q, \mathbb{R}^d)$) to study $\overline{\mathrm{Emb}}_c(M_n, \mathbb{R}^d)$. They show that the kth approximation of the Taylor tower associated to the chain complex $S_*(\overline{\mathrm{Emb}}_c(\mathbb{R}^q, \mathbb{R}^d))$, that is the Taylor tower of the functor $V \longmapsto S_*(\overline{\mathrm{Emb}}_c(V, \mathbb{R}^d))$ can be expressed in terms of morphisms between infinitesimal bimodules over the operad $S_*(B_q)$. The goal of this section is to obtain similar results (for the Taylor tower of $\overline{\mathrm{Emb}}_c(M_n, \mathbb{R}^d)$) as them. To state and prove our results, it is easiest to first review what is done in [2]. Let us start with *infinitesimal bimodules.*

5.2.1 Infinitesimal bimodules

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category.

Definition 5.2.4. *For an operad $\mathcal{O}$ in $\mathcal{C}$, a* right module *over $\mathcal{O}$ is a symmetric sequence $P = \{P(r)\}_{r\geq 0}$ in $\mathcal{C}$ endowed with insertions maps*

$$\circ_i \colon P(r) \otimes \mathcal{O}(t) \longrightarrow P(r+t-1), 1 \leq i \leq r,$$

satisfying the following three axioms

(RM)$_1$: For $1 \leq i \leq s$ and $1 \leq j \leq r$, the following diagram commutes

$$\begin{tikzcd}
P(r) \otimes \mathcal{O}(s) \times \mathcal{O}(t) \arrow[r, "id\otimes\circ_i"] \arrow[d, "\circ_j\otimes id"'] & P(r) \otimes \mathcal{O}(s+t-1) \arrow[d, "\circ_j"] \\
P(r+s-1) \otimes \mathcal{O}(t) \arrow[r, "\circ_{j-1+i}"] & P(r+s+t-2).
\end{tikzcd}$$

(RM)$_2$: For $1 \leq j < i \leq r$, the following diagram commutes

$$\begin{tikzcd}
P(r) \otimes \mathcal{O}(s) \otimes \mathcal{O}(t) \arrow[r, "id\otimes T"] \arrow[dd, "\circ_j\otimes id"'] & P(r) \otimes \mathcal{O}(t) \otimes \mathcal{O}(s) \arrow[d, "\circ_i\otimes id"] \\
& P(r+t-1) \otimes \mathcal{O}(s) \arrow[d, "\circ_j"] \\
P(r+s-1) \otimes \mathcal{O}(t) \arrow[r, "\circ_{i-1+s}"] & P(r+s+t-2).
\end{tikzcd} \tag{5.2.3}$$

Here $T\colon \mathcal{O}(s)\otimes\mathcal{O}(t)\xrightarrow{\cong}\mathcal{O}(t)\otimes\mathcal{O}(s)$ *is the isomorphism coming from the symmetric structure of* $\mathcal{C}$*.*

(RM)$_3$*: For* $1\leq i\leq r$*, the composite*

$$P(r)\xrightarrow{\cong}P(r)\otimes\mathbf{1}\xrightarrow{id\otimes\eta}P(r)\otimes\mathcal{O}(1)\xrightarrow{\circ_i}P(r)$$

is the identity. Here $\eta\colon \mathbf{1}\longrightarrow\mathcal{O}(1)$ *is the unit of the operad* $\mathcal{O}$*.*

Definition 5.2.5. *For an operad* $\mathcal{O}$ *in* $\mathcal{C}$*, a* weak left module *over* $\mathcal{O}$ *is a symmetric sequence* $P=\{P(r)\}_{r\geq 0}$ *in* $\mathcal{C}$ *endowed with insertions maps*

$$\circ_i\colon \mathcal{O}(r)\otimes P(t)\longrightarrow P(r+t-1), 1\leq i\leq r,$$

satisfying the following two axioms

(WLM)$_1$*: For* $1\leq i\leq r$ *and* $1\leq j\leq s$*, the following diagram commutes*

$$\begin{array}{ccc}\mathcal{O}(r)\otimes\mathcal{O}(s)\otimes P(t) & \xrightarrow{\circ_i\otimes id} & \mathcal{O}(r+s-1)\otimes P(t)\\ \downarrow{\scriptstyle id\otimes\circ_j} & & \downarrow{\scriptstyle \circ_{i-1+j}}\\ \mathcal{O}(r)\otimes P(s+t-1) & \xrightarrow{\circ_i} & P(r+s+t-2).\end{array}\tag{5.2.4}$$

(WLM)$_2$*: The following composition is the identity*

$$P(r)\xrightarrow{\cong}\mathbf{1}\otimes P(r)\xrightarrow{\eta\otimes id}\mathcal{O}(1)\otimes P(r)\xrightarrow{\circ_1}P(r)\ .$$

Notice that in general the notion of a weak left module is different to the standard one of a left module. In fact, a left module structure does not always imply a weak left module structure. This is because a left module over $\mathcal{O}$ is usually defined as a symmetric sequence $P=\{P(r)\}_{r\geq 0}$ in $\mathcal{C}$ equipped with structure morphisms

$$\mathcal{O}(k)\otimes P(i_1)\otimes\cdots\otimes P(i_k)\longrightarrow P(i_1+\cdots+i_k),$$

and the sequence P does not contain in general the unit element $\mathbf{1}\longrightarrow P(1)$.

Definition 5.2.6. *For an operad* $\mathcal{O}$ *in* $\mathcal{C}$*, a* weak bimodule *or* infinitesimal bimodule *over* $\mathcal{O}$ *is a symmetric sequence* $P=\{P(r)\}_{r\geq 0}$ *in* $\mathcal{C}$ *endowed with a right module and a weak left module structures over* $\mathcal{O}$ *that are compatible in the sense that the following two axioms hold*

(IB)$_1$*: The diagram obtained from (5.2.4) by replacing* $\mathcal{O}(s)$ *by* $P(s)$*,* $P(t)$ *by* $\mathcal{O}(t)$*, and* $\mathcal{O}(r+s-1)$ *by* $P(r+s-1)$ *commutes.*

(IB)$_2$*: The diagram obtained from (5.2.3) by replacing* $P(r)$ *by* $\mathcal{O}(r)$*,* $\mathcal{O}(t)$ *by* $P(t)$*, and* $P(r+s-1)$ *by* $\mathcal{O}(r+s-1)$ *commutes.*

Let $\underset{\mathcal{O}}{\mathrm{InfBim}}$ denote the category of infinitesimal bimodules over an operad $\mathcal{O}$, and let $\underset{\mathcal{O}}{\mathrm{InfBim}}_{\leq k}$ denote its kth truncation. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two infinitesimal bimodules over $\mathcal{O}$, we denote by $\mathrm{h}\underset{\mathcal{O}}{\mathrm{InfBim}}(\mathcal{B}_1,\mathcal{B}_2)$ the derived object of infinitesimal bimodules morphisms from $\mathcal{B}_1$ to $\mathcal{B}_2$.

Example 5.2.7. *Let $g\colon \mathcal{O} \longrightarrow \mathcal{P}$ be a morphism in the category of operads. Then $\mathcal{P}$ is equipped with a canonical structure of infinitesimal bimodule over $\mathcal{O}$. The weak left module structure is given by the composite*

$$\mathcal{O}(r)\otimes\mathcal{P}(s) \xrightarrow{g_r\otimes id} \mathcal{P}(r)\otimes\mathcal{P}(s) \xrightarrow{\circ_i} \mathcal{P}(r+s-1)\ ,$$

and the right one by

$$\mathcal{P}(r)\otimes\mathcal{O}(s) \xrightarrow{id\otimes g_s} \mathcal{P}(r)\otimes\mathcal{P}(s) \xrightarrow{\circ_i} \mathcal{P}(r+s-1)\ .$$

Remark 5.2.8. *Let $\mathcal{A}s$ be the associative operad in $\mathcal{C}$ (it was introduced in Example 1.4.11). It is not difficult to see that an infinitesimal bimodule over $\mathcal{A}s$ is a cosimplicial object $\mathcal{O}^\bullet$ in $\mathcal{C}$ (notice that the converse is not true in general). For $1\leq i\leq p$, the cofaces morphisms $d^i\colon \mathcal{O}^p \longrightarrow \mathcal{O}^{p+1}$ are defined using the right module structure ($d^i(x) = x\circ_i \mu$, where $\mu\colon \mathbf{1}\longrightarrow \mathcal{A}s(2)$ is the special operation in arity 2), and for $i\in\{0,p+1\}$, d^i is defined using the weak left module structure ($d^0(x)=\mu\circ_2 x$ and $d^{p+1}(x)=\mu\circ_1 x$). The codegeneracies morphisms $s^j : \mathcal{O}^{p+1}\longrightarrow \mathcal{O}^p$ are defined using the right module structure ($s^j(y) = y\circ_j e$, where $e\colon \mathbf{1}\longrightarrow\mathcal{A}s(0)$ is the special operation in arity 0).*

Let $\mathcal{O}$ be a multiplicative operad in topological spaces Top, that is, there exists a map $\mathcal{A}s \longrightarrow \mathcal{O}$ from the associative operad to $\mathcal{O}$. Then, by Example 5.2.7, the sequence $\{\mathcal{O}(n)\}_{n\geq 0}$ is endowed with an infinitesimal bimodule structure over $\mathcal{A}s$. Therefore, by Remark 5.2.8, we have an associated cosimplicial object $\mathcal{O}^\bullet$ in $\mathcal{C}$. Notice that the homotopy totalization $\mathrm{hoTot}\mathcal{O}^\bullet$ (that we have introduced in Definition 1.2.11) can also be expressed in terms of derived morphisms between infinitesimal bimodules over $\mathcal{A}s$. That is,

$$\begin{aligned}\mathrm{hoTot}\mathcal{O}^\bullet = \underset{\mathrm{cTop}}{\mathrm{Nat}}\,(\widetilde{\Delta}^\bullet,\mathcal{O}^\bullet) \simeq \underset{\mathrm{cTop}}{\mathrm{hNat}}(\widetilde{\Delta}^\bullet,\mathcal{O}^\bullet) &\simeq \underset{\mathrm{cTop}}{\mathrm{hNat}}(\mathcal{A}s^\bullet,\mathcal{O}^\bullet)\\ &\simeq \underset{\mathcal{A}s}{\mathrm{hInfBim}}(\mathcal{A}s,\mathcal{O}).\end{aligned} \tag{5.2.5}$$

Here $\mathrm{hNat}(-,-)$ denotes the space of derived natural transformations.
In the case of Kontsevich's operad $\mathcal{K}_d$, which is a multiplicative operad by Proposition 2.2.8, its weak left module structure encodes the fact adding a point at the infinity ($-\infty$ for d^0 and $+\infty$ for d^{p+1}), and its right module structure encodes the fact doubling a point. Recall that $\mathcal{K}_d(p)$ is define to be the Kontsevich compactification (see Definition 2.2.2) of the configuration space of p points in $\mathbb{R}^d$. By (5.2.5), we have

$$\mathrm{hoTot}\mathcal{K}_d^\bullet \simeq \underset{\mathcal{A}s}{\mathrm{hInfBim}}(\mathcal{A}s,\mathcal{O}). \tag{5.2.6}$$

One can view an infinitesimal bimodule as a contravariant functor from a certain category. To be more precise, let us first give the following definition.

Definition 5.2.9. *We define Γ to be the category of finite pointed sets $r_+ = \{1,\cdots,r,*\}$ whose morphisms are maps preserving the base point $*$.*

For two objects r_+ and s_+ in Γ, we write $\Gamma(r_+, s_+)$ for the set of morphisms in Γ from r_+ to s_+. Define now (for an operad $\mathcal{O}$ in $\mathcal{C}$) an enriched category over $\mathcal{C}$, $\widetilde{\Gamma}(\mathcal{O})$, as follows. The objects of $\widetilde{\Gamma}(\mathcal{O})$ are finite pointed sets r_+. If r_+ and s_+ are two objects in $\widetilde{\Gamma}(\mathcal{O})$, then

$$\hom_{\widetilde{\Gamma}(\mathcal{O})}(r_+, s_+) = \bigoplus_{f \in \Gamma(r_+, s_+)} \bigotimes_{x \in s_+} \mathcal{O}(f^{-1}(x)).$$

Proposition 5.2.10. [2, Proposition 2.15] *Let $\mathcal{C}$ be a symmetric monoidal category, and let $\mathcal{O}$ be an operad in $\mathcal{C}$. Then the category of contravariant functors from $\widetilde{\Gamma}(\mathcal{O})$ to $\mathcal{C}$ is equivalent to the category of infinitesimal bimodules over $\mathcal{O}$.*

Let us make two remarks.

Remark 5.2.11. *Let $Com = \{*\}_{r \geq 0}$ be the commutative symmetric operad in topological spaces. Then it is easy to see that $\widetilde{\Gamma}(Com) = \Gamma$. Therefore, by Proposition 5.2.10, an infinitesimal bimodule over Com is the same thing as a contravariant functor from Γ to* Top.

Before making the second remark concerning $\widetilde{\Gamma}(B_q)$, where B_q is of course the little q-disks operad, we first recall some necessary definitions. A self homeomorphism of $\mathbb{R}^q$ is said to be a *standard isomorphism* if it is the composition of a translation and a multiplication by a positive scalar. Let X be a subset of $\mathbb{R}^q$. We will say that a map $f\colon X \longrightarrow \mathbb{R}^q$ is a *standard embedding* if it is the composition of the inclusion and a standard isomorphism of $\mathbb{R}^q$. Given another subset Y of $\mathbb{R}^q$, the map $f\colon X \longrightarrow Y$ is called a *standard embedding* if $f\colon X \longrightarrow \mathbb{R}^q$ is a standard embedding, and its image lies in Y. We denote by $\mathrm{sEmb}(X, Y)$ the space of standard embeddings from X to Y. When $X = D^q$, the unit open ball of $\mathbb{R}^q$, the image $f(D^q)$ is called a *standard ball* in Y, and the complement $Y \backslash \overline{f(D^n)}$ is called a *standard antiball* in Y. The following remark gives an explicit description of the category $\widetilde{\Gamma}(B_q)$.

Remark 5.2.12. [2, Proposition 4.9] *The category $\widetilde{\Gamma}(B_q)$ is equivalent to the category whose*

- *an object is a finite pointed set r_+, which is viewed as r standard balls with one standard antiball,*
- *the space of morphisms from r_+ to s_+ is*

$$\hom_{\widetilde{\Gamma}(B_q)}(r_+, s_+) = \mathrm{sEmb}\left((\coprod_{i=1}^{r} D^q) \coprod (\mathbb{R}^q \backslash \overline{D^q}), (\coprod_{i=1}^{s} D^q) \coprod (\mathbb{R}^q \backslash \overline{D^q}) \right).$$

We end this section with examples of infinitesimal bimodules that will be used in the following sections. Let

$$\mathrm{sEmb}(-, Y)\colon \widetilde{\Gamma}(B_q) \longrightarrow \mathrm{Top}$$

be a functor defined by $\mathrm{sEmb}(-, Y)(U) = \mathrm{sEmb}(U, Y)$. It is very easy to see that it is a contravariant functor. This implies (by Proposition 5.2.10 above) the following example.

Example 5.2.13. *The collection*

$$\mathrm{sEmb}(-,Y) := \{\mathrm{sEmb}(\coprod_{i=1}^{r} D^q)\coprod(\mathbb{R}^q\backslash\overline{D^q}),Y)\}_{r\geq 0}$$

is an infinitesimal bimodule over B_q.

Example 5.2.14. *Let $(X,*)$ be a pointed space, and let $X^{\times -}\colon \Gamma \longrightarrow \mathrm{Top}$ be the functor defined by*

$$X^{\times -}(r_+) = \mathrm{Map}_*(r_+,X) = X^{\times r} = \underbrace{X\times\cdots\times X}_{r}.$$

This functor turns out to be a contravariant functor, and therefore (by Proposition 5.2.10 and Remark 5.2.11) the collection $\{X^{\times r}\}_{r\geq 0}$ is an infinitesimal bimodule over $Com = \{*\}_{n\geq 0}$.

Example 5.2.15. *For a symmetric monoidal functor $F\colon \mathcal{C}\longrightarrow\mathcal{D}$, for an operad $\mathcal{O}$ in $\mathcal{C}$, and for an infinitesimal bimodule $P=\{P(r)\}_{r\geq 0}$ over $\mathcal{O}$, the image $F(P)=\{F(P(r))\}_{r\geq 0}$ is obviously an infinitesimal bimodule over $F(\mathcal{O})$. Hence, $S_*(B_d)$ and $S_*(\mathrm{sEmb}(-,Y))$ are infinitesimal bimodules over $S_*(B_q)$, and $S_*(X^{\times -})$ is an infinitesimal bimodule over $\mathrm{Com}=\{S_*(*)\}_{n\geq 0}$. Recall that $S_*(-)$ is the normalized singular chain functor.*

Example 5.2.16. *Since the associative operad $\mathcal{A}s$ is the terminal object in the category of topological operads, it follows that there is a unique morphism $B_d\longrightarrow \mathcal{A}s$. Applying the normalized singular chain functor $S_*(-)$, we obtain a morphism $S_*(B_d)\longrightarrow \mathrm{Com}=\{S_*(*)\}_{n\geq 0}$. Moreover, there is a natural isomorphism $\mathrm{Com}\xrightarrow{\cong} H_0(B_d)$ (because $d\geq 2$ implies B_d connected). There is also an obvious morphism $H_0(B_d)\longrightarrow H_*(B_d)$. We thus get the sequence of morphisms*

$$S_*(B_d) \longrightarrow \mathrm{Com} \xrightarrow{\cong} H_0(B_d) \longrightarrow H_*(B_d)\ , \tag{5.2.7}$$

which endowed $H_*(B_d)$ with an infinitesimal bimodule structure over $S_*(B_d)$, and with an infinitesimal bimodule structure over Com by Example 5.2.7.

5.2.2 Review of the Taylor tower associated to $\overline{\mathbf{Emb}}_c(\mathbb{R}^q,\mathbb{R}^d)$

Here we recall some results of [2]. For more details, the reader can refer there.

Let $\mathcal{O}(\mathbb{R}^q)$ be the poset of open subsets of $\mathbb{R}^q$. Define the category $\mathcal{O}^c(\mathbb{R}^q)$ to be the subcategory of $\mathcal{O}(\mathbb{R}^q)$ whose objects are the complement of compact subsets of $\mathbb{R}^q$. Define also the category $\mathcal{O}^c_k(\mathbb{R}^q)$ to be the subcategory of $\mathcal{O}^c(\mathbb{R}^q)$ consisting of disjoint unions $U=U_0\cup U_1$ such that U_0 is the complement of a closed ball, and U_1 is the disjoint union of at most k open balls in $\mathbb{R}^q$. Consider now the functor

$$\overline{\mathrm{Emb}}_c(-,\mathbb{R}^d)\colon \mathcal{O}^c(\mathbb{R}^q)\longrightarrow \mathrm{Top}$$

of compactly-supported embeddings. We are going to make the "compactly supported" version of Goodwillie-Weiss embedding calculus [57, 19] with that functor. First of all, its kth approximation

$$T_k\overline{\mathrm{Emb}}_c(-,\mathbb{R}^d)\colon \mathcal{O}^c(\mathbb{R}^q) \longrightarrow \mathrm{Top}$$

is defined by

$$T_k\overline{\mathrm{Emb}}_c(V,\mathbb{R}^d) = \underset{U\subseteq V, U\in\mathcal{O}_k^c(\mathbb{R}^q)}{\mathrm{holim}} \overline{\mathrm{Emb}}_c(U,\mathbb{R}^d).$$

One shows that $T_k\overline{\mathrm{Emb}}_c(-\mathbb{R}^d)$ can be expressed in terms of morphisms of infinitesimal bimodules over the little q-disks operad B_q. In the following proposition, there is $\mathrm{sEmb}(-,\mathbb{R}^q)$ in the place of $\overline{\mathrm{Emb}}_c(-,\mathbb{R}^q)$ because of [1, Proposition 7.1].

Proposition 5.2.17. [2, Theorem 5.10] or [51, Theorem 6.1] *For $d > q$ and $k \leq \infty$, we have the weak equivalences*

$$\begin{aligned} T_k\overline{\mathrm{Emb}}_c(\mathbb{R}^q,\mathbb{R}^d) &\simeq \underset{B_q}{\mathrm{hInfBim}_{\leq k}}(\mathrm{sEmb}(-,\mathbb{R}^q), B_d) \\ &\simeq \underset{B_q}{\mathrm{hInfBim}_{\leq k}}(B_q, B_d). \end{aligned}$$

Notice that a version of Proposition 5.2.17 was proved [5] by Boavida de Brito and Weiss (they develop the details of the proof of that proposition). The following remark says that the space of long knots can be expressed in terms of derived morphisms between infinitesimal bimodules.

Remark 5.2.18. *When $q = 1$ and $d > q + 2$, Proposition 5.2.17 gives another proof of Theorem 2.3.3 due to Sinha. Indeed, since the little p-disks operad B_p is weakly equivalent (for all $p \geq 1$) to the Kontsevich operad $\mathcal{K}_p$ by Theorem 2.2.9, and since $\mathcal{K}_1$ is the associative operad $\mathcal{A}s$ by Remark 2.2.7, we deduce the following weak equivalences in which the last one comes from (5.2.6)*

$$\overline{\mathrm{Emb}}_c(\mathbb{R}^1,\mathbb{R}^d) \simeq \underset{B_1}{\mathrm{hInfbim}}(B_1, B_d) \simeq \underset{\mathcal{A}s}{\mathrm{hInfbim}}(\mathcal{A}s, \mathcal{K}_d) \simeq \mathrm{hoTot}\mathcal{K}_d^{\bullet}.$$

Proposition 5.2.17 admits an algebraic version, which is obtained by considering the composite

$$\mathcal{O}^c(\mathbb{R}^q) \xrightarrow{\overline{\mathrm{Emb}}_c(-,\mathbb{R}^d)} \mathrm{Top} \xrightarrow{S_*(-)} \mathrm{Ch}_*$$

from the category $\mathcal{O}^c(\mathbb{R}^q)$ to chain complexes.

Proposition 5.2.19. [2, Proposition 5.13] *For $d > q$ and $k \leq \infty$ there are weak equivalences*

$$\begin{aligned} T_kS_*\overline{\mathrm{Emb}}_c(\mathbb{R}^q,\mathbb{R}^d) &\simeq \underset{S_*B_q}{\mathrm{hInfBim}_{\leq k}}(S_*\mathrm{sEmb}(-,\mathbb{R}^q), S_*B_d) \\ &\simeq \underset{S_*B_q}{\mathrm{hInfBim}_{\leq k}}(S_*B_q, S_*B_d). \end{aligned}$$

Let S^q be the q dimensional sphere, which is viewed as the one-point compactification of $\mathbb{R}^q$, that is $S^q = \mathbb{R}^q \cup \{\infty\}$, and which is pointed at ∞. By Example 5.2.15, the sequence of chain complexes $S_*((S^q)^{\times -}) = \{S_*((S^q)^{\times r})\}_{r\geq 0}$ is an infinitesimal bimodule over com. In the following proposition, the first weak equivalence is proved in [2, Proposition 6.1] and the second one in [2, Proposition 7.3].

Proposition 5.2.20. [2, Proposition 6.1 and Proposition 7.3] *For $d \geq 2q+1$ and $k \leq \infty$, we have the following weak equivalences*

-
$$T_k S_* \overline{\mathrm{Emb}}_c(\mathbb{R}^q, \mathbb{R}^d) \simeq \underset{S_* B_q}{\mathrm{hInfBim}_{\leq k}}(S_* B_q, H_*(B_d; \mathbb{Q})$$

-
$$T_k S_* \overline{\mathrm{Emb}}_c(\mathbb{R}^q, \mathbb{R}^d) \simeq \underset{\mathrm{Com}}{\mathrm{hInfBim}_{\leq k}}(S_*((S^q)^{\times -}), H_*(B_d; \mathbb{Q})).$$

5.2.3 The Taylor tower associated to $\overline{\mathbf{Emb}}_c(N, \mathbb{R}^d)$

In [2] Arone and Turchin study the space $\overline{\mathrm{Emb}}(N, \mathbb{R}^d)$ of all smooth embeddings of an open submanifold $N \subseteq \mathbb{R}^q$ in $\mathbb{R}^d$. They prove that kth approximation of the Taylor tower associated to $\overline{\mathrm{Emb}}(N, \mathbb{R}^d)$ can be expressed in terms of derived morphisms between right modules over B_q. In this section we look at the space $\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d)$ of *compactly supported embeddings*, where N is the complement of a compact subset of $\mathbb{R}^q$. The goal here is to show that similar results as those mentioned in Section 5.2.2 hold for that space. Further in Section 5.3, we will apply (in order to prove Theorem 5.1.3) the results of this section with $N = M = M_1$, and further in Section 5.5 we will apply them with $N = M_n$ (see (5.2.2) for the definition of M_n).

Let us begin with the definition of $\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d)$. Let $K \subseteq \mathbb{R}^q$ be a compact subset, let $D = B(x, \delta) \subseteq \mathbb{R}^q$ be an open ball containing K, and let $\epsilon\colon \mathbb{R}^q \hookrightarrow \mathbb{R}^d$ be a fixed linear embedding defined by $\epsilon(t) = (0, \cdots, 0, -t_1, \cdots, -t_q)$ where $t = (t_1, \cdots, t_q)$. Define N to be the complement of K. That is,

$$N = \mathbb{R}^q \backslash K.$$

In the following definition $\overline{D}$ is of course the closed ball associated to D.

Definition 5.2.21.
- *A* long embedding *, $q \geq 1$ and $d \geq 3$, is a smooth embedding $f\colon \mathbb{R}^q \hookrightarrow \mathbb{R}^d$ satisfying the boundary conditions

$$\begin{cases} f(\overline{D}) \subseteq \mathbb{R}^{d-q} \times \overline{D} & \\ f(t) = \epsilon(t) & \text{if} \quad t \notin D \end{cases}$$

- *Given a long embedding $f\colon \mathbb{R}^q \hookrightarrow \mathbb{R}^d$, the image $f(N)$ is called a* compactly supported embedding *of N in $\mathbb{R}^d$.*

Notice that when $D =]-1, 1[$ and $q = 1$, Definition 5.2.21 coincides with Definition 2.1.1 of a long knot. Let $\mathrm{Emb}_c(N, \mathbb{R}^d)$ denote the space of compactly supported embeddings of N inside $\mathbb{R}^d$ (as in the case of long knots and long links, it is equipped with the weak $\mathcal{C}^\infty$-topology). Define the space $\mathrm{Imm}_c(N, \mathbb{R}^d)$ of compactly supported immersions analogously.

Definition 5.2.22. *The space* $\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d)$ *is defined to be the homotopy fiber of the inclusion*

$$\mathrm{Emb}_c(N, \mathbb{R}^d) \hookrightarrow \mathrm{Imm}_c(N, \mathbb{R}^d).$$

By abuse of terminology, we call it the space of compactly supported embeddings.

Notice that when $N = M$ the space $\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d)$ is weakly equivalent to the space of long links (this has been proved in Proposition 5.2.2), which is the space we study in this chapter.
Define now $\mathcal{O}(N)$ to be the category whose objects are open subsets of N and morphisms are inclusions. Define also the categories $\mathcal{O}^c(N)$ and $\mathcal{O}^c_k(N)$ as follows.

Definition 5.2.23.
- *We define $\mathcal{O}^c(N)$ to be the subcategory of $\mathcal{O}(N)$ the objects of which are the complements of compact subsets in N.*
- *We define $\mathcal{O}^c_k(N)$ to be the subcategory of $\mathcal{O}^c(N)$ consisting of $U = V \cup W$ such that*
  - $V \cap W = \emptyset$;
  - *V is the complement of a closed ball in N;*
  - *W is the disjoint union of at most k open balls.*

Taking the functors

$$\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)\colon \mathcal{O}^c(N) \longrightarrow \mathrm{Top} \qquad \text{and} \qquad S_*\overline{\mathrm{Emb}}_c(-, \mathbb{R}^d)\colon \mathcal{O}^c(N) \longrightarrow \mathrm{Ch}_*$$

as inputs in Goodwillie-Weiss embedding calculus, we have the following two propositions.

Proposition 5.2.24. *For $d > q$ and $k \leq \infty$ there is a weak equivalence*

$$T_k\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d) \simeq \underset{B_q}{\mathrm{hInfBim}_{\leq k}}(\mathrm{sEmb}(-, N), B_d).$$

Proof. The proof works exactly in the similar way as that of the first weak equivalence of Proposition 5.2.17. □

Proposition 5.2.25. *For $d > q$ and $k \leq \infty$ there is a weak equivalence*

$$T_kS_*\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d) \simeq \underset{S_*B_q}{\mathrm{hInfBim}_{\leq k}}(S_*\mathrm{sEmb}(-, N), S_*B_d).$$

Proof. Here the proof also works exactly in the similar way as that of the first equivalence of Proposition 5.2.19. □

Remark 5.2.26. *In Proposition 5.2.24 we don't have the equivalence*

$$\underset{B_q}{\mathrm{hInfBim}_{\leq k}}(\mathrm{sEmb}(-, N), B_d) \simeq \underset{B_q}{\mathrm{hInfBim}_{\leq k}}(B_q, B_d)$$

as in Proposition 5.2.17. This is because $\mathrm{sEmb}(-, N)$ *is not weakly equivalent to B_q for general N. Obviously, the same remark holds for Proposition 5.2.25.*

Applying now the relative formality Theorem 2.4.3 (which says that for $d \geq 2q+1$ the inclusion $B_q \hookrightarrow B_d$ is $\mathbb{R}$-formal) to Proposition 5.2.25, we obtain the following proposition.

Proposition 5.2.27. *For $d \geq 2q+1$ and $k \leq \infty$, there is a weak equivalence*

$$T_k S_*(\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d); \mathbb{Q}) \simeq \underset{S_* B_q}{\mathrm{hInfBim}}_{\leq k}(S_*(\mathrm{sEmb}(-, N); \mathbb{Q}), H_*(B_d; \mathbb{Q})).$$

Proof. The proof is the same as that of [2, Proposition 6.1]. The idea is to prove the assertion which states that the zigzag of quasi-isomorphisms connecting $S_*(B_q; \mathbb{R}) \longrightarrow S_*(B_d; \mathbb{R})$ to $H_*(B_q; \mathbb{R}) \longrightarrow H_*(B_d; \mathbb{R})$ holds in the category $\underset{S_* B_q}{\mathrm{hInfBim}}$ of derived infinitesimal bimodules over $S_* B_q$. This assertion is proved using essentially the general construction that associates to a morphism $g \colon \mathcal{O} \longrightarrow \mathcal{P}$ of operads in $\mathcal{C}$ (here we assume that $\mathcal{C}$ is a symmetric monoidal model category may be cofibrantly generated) a pair

$$\widetilde{\mathrm{ind}} \colon \underset{\mathcal{O}}{\mathrm{hInfBim}} \rightleftarrows \underset{\mathcal{P}}{\mathrm{hInfBim}} : \mathrm{res} \tag{5.2.8}$$

of adjoint functors ($\widetilde{\mathrm{ind}}$ being the derived left adjoint). Here res is just the restriction functor , and $\widetilde{\mathrm{ind}}$ is the induction functor defined as the derived left Kan extension

$$\begin{tikzcd} \widetilde{\Gamma}(\mathcal{O}) \arrow[r] \arrow[d, "\widetilde{\Gamma}(g)"'] & \mathcal{C} \\ \widetilde{\Gamma}(\mathcal{P}) \arrow[ur, dotted] & \end{tikzcd}$$

along the functor $\widetilde{\Gamma}(g) \colon \widetilde{\Gamma}(\mathcal{O}) \longrightarrow \widetilde{\Gamma}(\mathcal{P})$ induced by g. Recall that by Proposition 5.2.10, the category $\underset{\mathcal{O}}{\mathrm{InfBim}}$ is equivalent to the category of contravariant functors from $\widetilde{\Gamma}(\mathcal{O})$ to $\mathcal{C}$, similarly to the category $\underset{\mathcal{P}}{\mathrm{InfBim}}$. $\square$

Let $\widehat{N}$ be the one-point compactification of N pointed at ∞,

$$\widehat{N} = N \cup \{\infty\}.$$

Recalling that $S_*(\widehat{N}^{\times -})$ and $H_*(B_d; \mathbb{Q})$ are infinitesimal bimodules over Com from Example 5.2.15 and Example 5.2.16, we have the following proposition which is proved in the similar way as the second part of Proposition 5.2.20.

Proposition 5.2.28. *For $d \geq 2q+1$ and $k \leq \infty$, there is a weak equivalence*

$$T_k S_*(\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d); \mathbb{Q}) \simeq \underset{\mathrm{Com}}{\mathrm{hInfBim}}_{\leq k}(S_*(\widehat{N}^{\times -}), H_*(B_d; \mathbb{Q})).$$

Proof. Let $g \colon B_q \longrightarrow \mathrm{Com} = \{*\}_{n \geq 0}$ be the unique morphism of operads from B_q to Com. By (5.2.8) the morphisms g and $S_*(g) \colon S_* B_q \longrightarrow S_*(\mathrm{Com}) = \mathrm{Com}$ induce respectively the pairs

$$\widetilde{\mathrm{ind}} \colon \underset{B_q}{\mathrm{hInfBim}} \rightleftarrows \underset{\mathrm{Com}}{\mathrm{hInfBim}} : \mathrm{res} \tag{5.2.9}$$

and

$$\widetilde{\mathrm{ind}}\colon \underset{S_*B_q}{\mathrm{hInfBim}} \rightleftarrows \underset{\mathrm{Com}}{\mathrm{hInfBim}} : \mathrm{res} \tag{5.2.10}$$

of adjoint functors. Notice that by the sequence (5.2.7), the restriction functor sends H_*B_q (viewed as an infinitesimal bimodule over Com) to itself (viewed as an infinitesimal bimodule over S_*B_q). That is,

$$\mathrm{res}(H_*B_q) = H_*B_q. \tag{5.2.11}$$

Since we have the following weak equivalences (the first one comes from Proposition 5.2.27, and the second one from the adjunction (5.2.10))

$$\begin{aligned} T_kS_*\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d) &\simeq \underset{S_*B_q}{\mathrm{hInfBim}_{\leq k}}(S_*\mathrm{sEmb}(-,N), H_*B_d) \\ &= \underset{S_*B_q}{\mathrm{hInfBim}_{\leq k}}(S_*\mathrm{sEmb}(-,N), \mathrm{res}(H_*B_d)) \text{ by (5.2.11)} \\ &\cong \underset{\mathrm{Com}}{\mathrm{hInfBim}_{\leq k}}(\widetilde{\mathrm{ind}}(S_*\mathrm{sEmb}(-,N)), H_*B_d), \end{aligned}$$

to prove Proposition 5.2.28, it suffices to prove that the functors $S_*(\widehat{N}^{\times -})$ and $\widetilde{\mathrm{ind}}(S_*\mathrm{sEmb}(-,N))$ are weakly equivalent as infinitesimal bimodules over Com. Since the functor $\widetilde{\mathrm{ind}}(S_*\mathrm{sEmb}(-,N))$ is the homotopy colimit of a certain diagram, and since the singular chain functor $S_*(-)$ commutes with homotopy colimits, it suffices to prove that there is a weak equivalence

$$\widetilde{\mathrm{ind}}(\mathrm{sEmb}(-,N)) \simeq \widehat{N}^{\times -} \tag{5.2.12}$$

holding in the category $\underset{\mathrm{Com}}{\mathrm{hInfBim}}$. The rest of the proof is devoted to (5.2.12). Recall first that the functor $\widetilde{\mathrm{ind}}(\mathrm{sEmb}(-,N))$ is the homotopy left Kan extension

$$\begin{tikzcd} \widetilde{\Gamma}(B_q) \arrow[r, "\mathrm{sEmb}(-,N)"] \arrow[d, "\widetilde{\Gamma}(g)"'] & \mathcal{C} \\ \widetilde{\Gamma}(\mathrm{Com}) = \Gamma \arrow[ur, dotted, "\widetilde{\mathrm{ind}}(\mathrm{sEmb}(-,N))"'] & \end{tikzcd}$$

of $\mathrm{sEmb}(-,N)$ along $\widetilde{\Gamma}(g)$, and let us prove (5.2.12) with $N = U \in \widetilde{\Gamma}(B_q)$. In that case, $\mathrm{sEmb}(-,U)$ is the free functor generated by U (recall that by Remark 5.2.12, the object U can be viewed as a disjoint union of standard balls with one standard antiball), and therefore $\widetilde{\mathrm{ind}}(\mathrm{sEmb}(-,U))$ is the free functor generated by $\widetilde{\Gamma}(g)(U)$. That is,

$$\widetilde{\mathrm{ind}}(\mathrm{sEmb}(-,U)) \simeq \mathrm{Map}_*(-, \widetilde{\Gamma}(g)(U)). \tag{5.2.13}$$

Notice that (5.2.13) is natural in U. Notice also that $\mathrm{Map}_*(-, \widetilde{\Gamma}(g)(U))$ is not weakly equivalent to the functor $\mathrm{Map}_*(-,U)$ because the antiball of U is not contractible. To correct this, let us define $\widehat{U}$ to be the one-point compactification of U, that is $\widehat{U} = U \cup \{\infty\}$. Here the point ∞ is of course added to the antiball of U, and it is the base point of $\widehat{U}$. We now have the following weak equivalence

$$\mathrm{Map}_*(-, \widetilde{\Gamma}(g)(U)) \simeq \mathrm{Map}_*(-, \widehat{U}), \tag{5.2.14}$$

which is also natural in U. Combining (5.2.13) and (5.2.14), we get

$$\widetilde{\mathrm{ind}}(\mathrm{sEmb}(-, U)) \simeq \mathrm{Map}_*(-, \widehat{U}). \tag{5.2.15}$$

Let us consider now the general case $\mathrm{sEmb}(-, N)$. It is not difficult to see that this functor is the homotopy colimit

$$\mathrm{sEmb}(-, N) \simeq \operatorname*{hocolim}_{U} \mathrm{sEmb}(-, U).$$

Therefore we have the following weak equivalences (the first one comes from the fact that the functor $\widetilde{\mathrm{ind}}$ commutes with homotopy colimits)

$$\begin{aligned} \widetilde{\mathrm{ind}}(\mathrm{sEmb}(-, N)) &\simeq \operatorname*{hocolim}_{U} \widetilde{\mathrm{ind}}(\mathrm{sEmb}(-, U)) \\ &\simeq \operatorname*{hocolim}_{U} \mathrm{Map}_*(-, \widehat{U}) \text{ by (5.2.15)} \\ &\simeq \mathrm{Map}_*(-, \widehat{N}), \end{aligned}$$

thus completing the proof. □

5.3 A cosimplicial model for the singular chain complex of the space of long links

The goal of this section is to detail the proof of Theorem 5.1.3 announced in the introduction. Before doing that, we state some intermediate results. As in Section 5.2, the ground field is $\mathbb{Q}$ here.

Let us start with the definition of a *right* Γ*-module*. Recall from Definition 5.2.9 the category Γ. Pirashvili [37] defines a right Γ-module as a contravariant functor from Γ to vector spaces. For our purposes, it is defined as follows.

Definition 5.3.1. *A* right Γ-module *is a contravariant functor from* Γ *to chain complexes* Ch_*.

We denote by $\operatorname*{Rmod}_{\Gamma}$ the category of right Γ-modules. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two right Γ-modules, by $\operatorname*{hRmod}_{\Gamma}(\mathcal{M}_1, \mathcal{M}_2)$, we denote the derived chain complex of right Γ-modules morphisms from $\mathcal{M}_1$ to $\mathcal{M}_2$. Right Γ-modules are related to infinitesimal bimodules (that we have seen in Section 5.2.1) as is shown in the following lemma, which is just the algebraic version of Remark 5.2.11.

Lemma 5.3.2. *The category of infinitesimal bimodules over the commutative operad (in chain complexes) is equivalent to the category of right* Γ*-modules. That is,*

$$\operatorname*{InfBim}_{\mathrm{Com}} \cong \operatorname*{Rmod}_{\Gamma}.$$

Here are two examples of right Γ-modules that will be used in this section and in Section 5.4.

Example 5.3.3. *(i) The homology $H_*(B_d)\colon \Gamma \longrightarrow \mathrm{Ch}_*$ defined by*

$$H_*(B_d)(r_+) = H_*(B_d(r)),$$

where $r_+ = \{1, \cdots, r, *\}$, is a right Γ-module. This comes from Example 5.2.16 and Lemma 5.3.2.

(ii) Let $(X, *)$ be a pointed topological space. Then, by Example 5.2.15 and Lemma 5.3.2, the singular chain functor $S_*(X^{\times -})\colon \Gamma \longrightarrow \mathrm{Ch}_*$ defined by

$$S_*(X^{\times -})(r_+) = S_*(\hom_{\Gamma}(r_+, X)) \cong S_*(X^{\times r})$$

is a right Γ-module.

We are now going to properly define the cosimplicial chain complex $L_*^\bullet$, which appears in Theorem 5.1.3. From now and in the rest of this chapter, if $X_\bullet$ is a simplicial set, we will denote by X its geometric realization. Let us consider the simplicial model $(\vee_{i=1}^m S^1)_\bullet$ of the wedge $\vee_{i=1}^m S^1$ of m copies of the circle, which has a unique 0-simplex and exactly m non degenerate 1-simplices (it was well defined at the end of Section 1.2.1 from Chapter 1). This simplicial model is actually a simplicial object in Γ, where the base points are taken to be the 0-simplex and its degeneracies. Hence, we can form the composite $H_*(B_d)((\vee_{i=1}^m S^1)_\bullet)\colon \Delta \longrightarrow \mathrm{Ch}_*$, which yields a cosimplicial chain complex.

Definition 5.3.4. *The cosimplicial chain complex $L_*^\bullet$ is defined to be the composite $H_*(B_d)((\vee_{i=1}^m S^1)_\bullet)$,*

$$L_*^\bullet = H_*(B_d)((\vee_{i=1}^m S^1)_\bullet).$$

Definition 5.3.5. *Let $X_\bullet$ be a simplicial object in Γ, and let A be a right Γ-module. Then the composite $A(X_\bullet)\colon \Delta \longrightarrow \mathrm{Ch}_*$ is a cosimplicial chain complex, and the homology of its totalization is called the X-*homology *of A.*

The following proposition is known to specialists, but its proof is written nowhere in my knowledge.

Proposition 5.3.6. *Consider the data of the previous definition. Then there is a weak equivalence of chain complexes*

$$\mathrm{Tot} A(X_\bullet) \simeq \mathrm{hRmod}_{\Gamma}(S_*(|X_\bullet|^{\times -}), A). \tag{5.3.1}$$

Proof. We will work with a field $\mathbb{K}$ of characteristic 0. For a set S we denote by $\mathbb{K}[S]$ the vector space generated by S, which will be viewed as a chain complex concentrated in degree 0.
We begin the proof by showing that there is an isomorphism

$$S_*\left(\left|X_\bullet^{\times -}\right|\right) \cong S_*(|X_\bullet|^{\times -}). \tag{5.3.2}$$

of right Γ-modules. To do that, let us consider the pair of functors

$$\Gamma \underset{\left|X_\bullet^{\times -}\right|}{\overset{|X_\bullet|^{\times -}}{\rightrightarrows}} \mathrm{Top}.$$

It is well known [28, Theorem 14.3](since the simplicial set $X_\bullet$ is countable) that there is an isomorphism

$$|X_\bullet| \times |X_\bullet| \xleftarrow{\cong} |X_\bullet \times X_\bullet|$$

in the category of topological spaces, and we can easily see that this isomorphism induces for each $p_+ \in \Gamma$ an isomorphism

$$\phi_{p_+} \colon |X_\bullet|^{\times p} \xleftarrow{\cong} \left|X_\bullet^{\times p}\right|,$$

which is natural in p_+. We thus get a natural isomorphism $\phi\colon |X_\bullet|^{\times -} \xleftarrow{\cong} \left|X_\bullet^{\times -}\right|$ and therefore, the isomorphism (5.3.2) holds in the category of right Γ-modules. From this latter isomorphism, we deduce the following one

$$\underset{\Gamma}{\mathrm{hRmod}}(S_*(|X_\bullet|^{\times -}), A) \cong \underset{\Gamma}{\mathrm{hRmod}}\left(S_*\left(\left|X_\bullet^{\times -}\right|\right), A\right). \tag{5.3.3}$$

In the second part of this proof, we are going to show (since the totalization $\mathrm{Tot}(A(X_\bullet))$ is weakly equivalent to the homotopy limit of the Δ-diagram $A(X_\bullet)$ in chain complexes) that the right hand side of (5.3.3) is quasi-isomorphism to the homotopy limit of a certain Δ-diagram. From now and in the rest of this proof, the standard simplicial set $\Delta^p_\bullet$ will be viewed as a simplicial object in Γ, where the base point of $\Delta^p_k = \underset{\Delta}{\hom}([k], [p])$ is taken to be the null morphism. We denote by $s\Gamma$ the category of simplicial objects in Γ, and by N the Dold-Kan normalization functor (it was defined in Section 1.5.1 from Chapter 1). Let us consider the pair of contravariant functors

$$\Gamma \underset{S_*\left(\left|X_\bullet^{\times -}\right|\right)}{\overset{\underset{[p]\in\Delta^{op}}{\mathrm{hocolim}}\left(\mathbb{K}[\underset{\Gamma}{\hom}(-, X_p)]\right)}{\rightrightarrows}} \mathrm{Ch}_*. \tag{5.3.4}$$

We want to build a natural weak equivalence between these two functors. So let $r_+ \in \Gamma$ be a finite pointed set. Then the simplicial structure of $X_\bullet$ induces a simplicial structure on $\underset{\Gamma}{\hom}(r_+, X_\bullet)$. By Yoneda's lemma, we have for each $p \geq 0$ the following isomorphism

$$\underset{s\Gamma}{\hom}(\Delta^p_\bullet, \underset{\Gamma}{\hom}(r_+, X_\bullet)) = \underset{s\Gamma}{\hom}(\underset{\Delta}{\hom}(\bullet, [p]), \underset{\Gamma}{\hom}(r_+, X_\bullet) \cong \underset{\Gamma}{\hom}(r_+, X_p),$$

which implies

$$\begin{aligned} \underset{[p]\in\Delta^{op}}{\mathrm{hocolim}}\left(\mathbb{K}[\underset{\Gamma}{\hom}(r_+, X_p)]\right) &\cong \underset{[p]\in\Delta^{op}}{\mathrm{hocolim}}\left(\mathbb{K}[\underset{s\Gamma}{\hom}(\Delta^p_\bullet, \underset{\Gamma}{\hom}(r_+, X_\bullet))]\right) \\ &\simeq NV_\bullet(r_+). \end{aligned} \tag{5.3.5}$$

Here $V_\bullet(r_+)$ is the simplicial chain complex defined by

$$V_p(r_+) = \mathbb{K}[\underset{s\Gamma}{\hom}(\Delta^p_\bullet, \underset{\Gamma}{\hom}(r_+, X_\bullet))].$$

Notice that the isomorphism and the weak equivalence of (5.3.5) are natural in r_+.

On the other hand, let $W_\bullet(r_+)$ be the simplicial chain complex defined by $W_p(r_+) = \mathbb{K}[\underset{\mathrm{Top}}{\hom}(\Delta^p, |X_\bullet^{\times r}|)]$. Then the associated chain complex (as defined at the beginning of Section 1.5.1) is nothing other than the singular chain complex $S_*\left(|X_\bullet^{\times r}|\right)$. Therefore, since the chain complex associated to a simplicial abelian group is quasi-isomorphic to its Dold-Kan normalization (see [16, Chapter III-Theorem 2.4]), there is a natural quasi-isomorphism

$$S_*\left(\left|X_\bullet^{\times r}\right|\right) \simeq NW_\bullet(r_+). \tag{5.3.6}$$

We have just defined a pair of contravariant functors

$$\Gamma \overset{NV_\bullet}{\underset{NW_\bullet}{\rightrightarrows}} \mathrm{Ch}_*.$$

Define now $\alpha_{r_+}\colon NV_p(r_+) \longrightarrow NW_p(r_+)$ by the formula $\alpha_{r_+}(f) = |f|$, where $f\colon \Delta^p_\bullet \longrightarrow \underset{\Gamma}{\hom}(r_+, X_\bullet)$ is a morphism in simplicial sets. It is straightforward to check that $\alpha\colon NV_\bullet \longrightarrow NW_\bullet$ is a quasi-isomorphism natural in r_+. This implies (with (5.3.5) and (5.3.6)) that there is a quasi-isomorphism

$$\underset{[p]\in\Delta^{op}}{\mathrm{hocolim}}\left(\mathbb{K}[\underset{\Gamma}{\hom}(-, X_p)]\right) \simeq S_*\left(\left|X_\bullet^{\times -}\right|\right). \tag{5.3.7}$$

in the category of right Γ-modules. We end the proof with the following summarizing in which the second line is because of (5.3.7)

$$\begin{array}{rcll}
\underset{\Gamma}{\mathrm{hRmod}}(C_*(|X_\bullet|^{\times -}), A) & \cong & \underset{\Gamma}{\mathrm{hRmod}}\left(S_*\left(\left|X_\bullet^{\times -}\right|\right), A\right) & \text{by (5.3.3)} \\
& \simeq & \underset{\Gamma}{\mathrm{hRmod}}\left(\underset{[p]\in\Delta^{op}}{\mathrm{hocolim}}\left(\mathbb{K}[\underset{\Gamma}{\hom}(-, X_p)]\right), A\right) & \\
& \simeq & \underset{[p]\in\Delta}{\mathrm{holim}}\left(\underset{\Gamma}{\mathrm{hRmod}}\left(\mathbb{K}[\underset{\Gamma}{\hom}(-, X_p)], A\right)\right) & \\
& \cong & \underset{[p]\in\Delta}{\mathrm{holim}}(A(X_p)) & \text{by Yoneda's lemma} \\
& \cong & \mathrm{Tot}(A(X_\bullet)). &
\end{array}$$

$\square$

Before starting the proof of Theorem 5.1.3, let us state the following theorem, which is proved by using Goodwillie-Weiss techniques for embedding calculus [58, 19].

Theorem 5.3.7. [58] *Let N be the complement of a compact subset of $\mathbb{R}^q$. Then, for $d > 2q+1$, the natural map*

$$S_*(\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d); \mathbb{Q}) \longrightarrow T_\infty S_*(\overline{\mathrm{Emb}}_c(N, \mathbb{R}^d); \mathbb{Q})$$

is a quasi-isomorphism.

Remark 5.3.8. *In the case $N = M \subseteq \mathbb{R}^2$ (see Definition 5.2.1), we have $q = 2$. We also have the following weak equivalence, which states that the pointed manifold $\widehat{M}$ is weakly equivalent to the wedge of m copies of the circle. That is,*

$$\widehat{M} \simeq \vee_{i=1}^{m} S^1.$$

We are now ready to prove Theorem 5.1.3, which states that $L_*^\bullet$ defined above (see Definition 5.3.4) is a cosimplicial model for the singular chain complex of the space $\overline{\mathcal{L}}_m^d$ of long links with m strings in $\mathbb{R}^d$.

Proof of Theorem 5.1.3. For $d > 5$, we have the following weak equivalences in which Theorem 5.3.7, Proposition 5.2.25, Proposition 5.2.27 and Proposition 5.2.28 are applied with $N = M$.

$$\begin{array}{rcl}
S_*(\overline{\mathcal{L}}_m^d) \otimes \mathbb{Q} & \simeq & S_*(\overline{\mathrm{Emb}}_c(M, \mathbb{R}^d); \mathbb{Q}) \qquad \text{by Proposition 5.2.2} \\
& \simeq & T_\infty S_*(\overline{\mathrm{Emb}}_c(M, \mathbb{R}^d); \mathbb{Q}) \qquad \text{by Theorem 5.3.7} \\
& \simeq & \underset{S_*B_2}{\mathrm{hInfBim}}(S_*\mathrm{sEmb}(-, M), S_*B_d) \qquad \text{by Proposition 5.2.25} \\
& \simeq & \underset{S_*B_2}{\mathrm{hInfBim}}(S_*\mathrm{sEmb}(-, M), H_*B_d) \qquad \text{by Proposition 5.2.27} \\
& \simeq & \underset{\mathrm{Com}}{\mathrm{hInfBim}}(S_*(\widehat{M}^{\times -}), H_*B_d) \qquad \text{by Proposition 5.2.28} \\
& \simeq & \underset{\Gamma}{\mathrm{hRmod}}(S_*(\widehat{M}^{\times -}), H_*B_d) \qquad \text{by Lemma 5.3.2} \\
& \simeq & \underset{\Gamma}{\mathrm{hRmod}}(S_*((\vee_{i=1}^m S^1)^{\times -}), H_*B_d) \qquad \text{since } \widehat{M} \simeq \vee_{i=1}^m S^1 \\
& \simeq & \mathrm{Tot} H_*B_d((\vee_{i=1}^m S^1)_\bullet) \qquad \text{by Proposition 5.3.6} \\
& = & \mathrm{Tot} L_*^\bullet \qquad \text{by Definition 5.3.4 of } L_*^\bullet.
\end{array}$$

□

5.4 Collapsing of the H_*BKSS associated to the Munson-Volić cosimplicial model $\mathcal{L}_m^{d\bullet}$

The goal of this section is to prove one of the main results of this thesis (Theorem 5.1.6), which states that the H_*BKSS associated to the Munson-Volić cosimplicial model for the space of long links collapses at the E^2 page rationally. As in the previous sections, the ground field here is $\mathbb{Q}$.

Let us start with a crucial Lemma 5.4.1. Before stating it, we recall that in [35] Munson and Volić build a cosimplicial space that gives a cosimplicial model for the space of long links of m strands in $\mathbb{R}^d$. Let us denote it by $\mathcal{L}_m^{d\bullet}$. It is built in the "same spirit" as the Sinha cosimplicial model $\mathcal{K}_d^\bullet$ (which was reviewed in Section 2.3) for the space of long knots. Besides that, one can easily check the following equality

$$\mathcal{L}_1^{d\bullet} = \mathcal{K}_d^\bullet.$$

When we say"in the same spirit", it means that the pth space $\mathcal{L}_m^{dp}$ is the Kontsevich compactification (see Definition 2.2.2) of the configuration space $\mathrm{Conf}(mp, \mathbb{R}^d)$, and cofaces and codegeneracies consist of doubling and forgetting some points. More precisely, the ith coface map consists of doubling simultaneously the ith point of each strand, while the jth codegeneracy map consists of forgetting simultaneously the jth point of each strand.

Recalling the notations $\{E^r(C_*^\bullet)\}_{r\geq 0}$ (where $C_*^\bullet$ is a cosimplicial chain complex), and $\{E^r(S_*(X^\bullet))\}_{r\geq 0}$ (where $X^\bullet$ is a cosimplicial space) from Section 1.5, we have the following lemma.

Lemma 5.4.1. *For $d \geq 3$, the E^1 pages of spectral sequences $\{E^r(L_*^\bullet)\}_{r\geq 0}$ and $\{E^r(S_*(\mathcal{L}_m^{d\bullet};\mathbb{Q}))\}_{r\geq 0}$ are isomorphic. That is,*

$$\{E^r(L_*^\bullet)\}_{r=1} \cong \{E^r(S_*(\mathcal{L}_m^{d\bullet};\mathbb{Q}))\}_{r=1}.$$

Before proving Lemma 5.4.1, recall that we have seen in Example 5.2.16 the Com-infinitesimal bimodule structure of $H_*(B_d)$. The homology $H_*(\mathcal{K}_d)$ of Kontsevich's operad $\mathcal{K}_d$ is endowed with a similar infinitesimal bimodule structure because of the sequence of morphisms

$$\mathrm{Com} = S_*(\mathcal{A}s) \xrightarrow{\cong} H_0(\mathcal{K}_d) \longrightarrow H_*(\mathcal{K}_d)\ .$$

As said before, $S_*(-)$ is the normalized singular chain functor. Recall also that a simplicial model $(\vee_{i=1}^m S^1)_\bullet$ of the wedge $\vee_{i=1}^m S^1$ of m copies of the circle has been provided at the end of Section 1.2.1.

Proof of Lemma 5.4.1. Since the diagram

$$\begin{tikzcd} H_*(B_d) \arrow[r, "\cong"] & H_*(\mathcal{K}_d) \\ H_0(B_d) \arrow[u] \arrow[r, "\cong"] & H_0(\mathcal{K}_d) \arrow[u] \end{tikzcd}$$

is commutative, it follows that the upper isomorphism (which is an immediate consequence of Theorem 2.2.9) holds in the category $\underset{\mathrm{Com}}{\mathrm{InfBim}}$. Therefore, since an infinitesimal bimodule over Com is the same thing as a right Γ-module (see Lemma 5.3.2), the same isomorphism ($H_*(B_d) \cong H_*(\mathcal{K}_d)$) holds in the category of right Γ-modules. This implies that the isomorphism

$$\begin{array}{rcll} L_*^\bullet & = & H_*(B_d)((\vee_{i=1}^m S^1)_\bullet) & \\ & \cong & H_*(\mathcal{K}_d)((\vee_{i=1}^m S^1)_\bullet) & \\ & = & H_*(\mathcal{L}_m^{d\bullet}) & \text{by Proposition 1.2.4 from Chapter 1} \end{array}$$

holds in the category of cosimplicial chain complexes, thus completing the proof. □

Lemma 5.4.2. *For $d \geq 3$ the spectral sequence $\{E^r(L_*^\bullet)\}_{r\geq 0}$ collapses at the E^2 page rationally.*

Proof. By Proposition 1.2.4 and Definition 5.3.4, we have $L_*^p = H_*(B_d(mp))$ for each $p \geq 0$. Since the homology $H_*(B_d(mp)$ is a chain complex with 0 differential, it follows that the vertical differential in the bicomplex associated to $L_*^\bullet$ is trivial. Therefore, the spectral sequence $\{E^r(L_*^\bullet)\}_{r\geq 0}$ collapses at the E^2 page. □

We are now ready to prove Theorem 5.1.6.

Proof of Theorem 5.1.6. The proof follows from the following three points:

- the E^1 pages of $\{E^r(L_*^\bullet)\}_{r\geq 0}$ and $\{E^r(S_*(\mathcal{L}_m^{d\bullet};\mathbb{Q}))\}_{r\geq 0}$ are isomorphic (by Lemma 5.4.1);
- for $d > 5$, the spectral sequences $\{E^r(L_*^\bullet)\}_{r\geq 0}$ and $\{E^r(S_*(\mathcal{L}_m^{d\bullet};\mathbb{Q}))\}_{r\geq 0}$ have the same abutment (by Theorem 5.1.3);
- the spectral sequence $\{E^r(L_*^\bullet)\}_{r\geq 0}$ collapses at the E^2 page (because of Lemma 5.4.2).

□

5.5 High dimensional analogues of spaces of long links

The goal of this short section is show that our method enables us to get the collapsing at the E^2 page of the spectral sequence computing the rational homology of he high dimensional analogues of spaces of long links.

Let us start with a definition.

Definition 5.5.1. *The* high dimensional analogues of spaces of long links *is the homotopy fiber of the inclusion*

$$\mathrm{Emb}(\coprod_{i=1}^{m}\mathbb{R}^n, \mathbb{R}^d) \hookrightarrow \mathrm{Imm}(\coprod_{i=1}^{m}\mathbb{R}^n, \mathbb{R}^d),$$

and it is denoted by $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^m \mathbb{R}^n, \mathbb{R}^d)$.

As in the case of long links, let us consider the cosimplicial chain complex

$$L_*^{n\bullet} := H_*(B_d, \mathbb{Q})((\vee_{i=1}^m S^n)_\bullet)$$

in which $(\vee_{i=1}^m S^n)_\bullet$ is the simplicial model (built in the similar way as $(\vee_{i=1}^m S^1)_\bullet$) of the wedge $\vee_{i=1}^m S^n$ of m copies of the n dimensional sphere S^n. The following theorem says that it gives a cosimplicial model for the singular chain complex $S_*\overline{\mathrm{Emb}}_c(\coprod_{i=1}^m \mathbb{R}^n, \mathbb{R}^d)$.

Theorem 5.5.2. *For $d > 2n+3$ there is a weak equivalence*

$$\mathrm{Tot}L_*^{n\bullet} \simeq S_*(\overline{\mathrm{Emb}}_c(\coprod_{i=1}^{m}\mathbb{R}^n, \mathbb{R}^d)) \otimes \mathbb{Q}.$$

Proof. The proof works exactly in the similar way as that of Theorem 5.1.3 given in Section 5.3. It suffices to replace M by M_n (recall that the open submanifold $M_n \subseteq \mathbb{R}^{n+1}$ was defined in equation (5.2.2)), B_2 by B_{n+1}, and of course S^1 by S^n and $L_*^\bullet$ by $L_*^{n\bullet}$, the rest being unchanged. □

The following corollary is a generalization of Corollary 5.1.4. It is also an immediate consequence of Theorem 5.5.2.

Corollary 5.5.3. *For $d > 2n+3$ there is an isomorphism*

$$H_*(\overline{\mathrm{Emb}}_c(\coprod_{i=1}^{m}\mathbb{R}^n, \mathbb{R}^d); \mathbb{Q}) \cong HH^{\vee_{i=1}^{m} S^n}(H_*(B_d; \mathbb{Q})).$$

Let us consider now the spectral sequence $\{E^r(L_*^{n\bullet})\}_{r\geq 0}$. It is clear (by Theorem 5.5.2) that it converges to the homology $H_*(\overline{\mathrm{Emb}}_c(\coprod_{i=1}^{m}\mathbb{R}^n, \mathbb{R}^d); \mathbb{Q})$, when $d > 2n+3$. We prove (exactly as Lemma 5.4.2 below) that this spectral sequence collapses at the E^2 page.

Proposition 5.5.4. *For $d > 2n+3$, the spectral sequence $\{E^r(L_*^{n\bullet})\}_{r\geq 0}$ computing the rational homology $H_*(\overline{\mathrm{Emb}}_c(\coprod_{i=1}^{m}\mathbb{R}^n, \mathbb{R}^d); \mathbb{Q})$ collapses at the E^2 page rationally.*

5.6 Poincaré series for the space of long links modulo m copies of long knots

The aim of this section is to prove that the radius of convergence, of the Poincaré series for the pair formed by the space of long links and the space of m copies of long knots, tends to 0 as m goes to the infinity. We also state a conjecture followed by a theorem concerning the radius of convergence for that pair. Here, the abreviation H^*BKSS means cohomology Bousfield-Kan spectral sequence.

Let us start by defining expressions that appear in the title of the section.

Definition 5.6.1. *Let X be a topological space.*

- *For $k \geq 0$ the k*th Betti number *, $b_k(X)$, of X is the rank of its kth homology group $H_k(X)$.*
- *The* Poincaré *series of X, denoted by $P_X[x]$, is the series $P_X[x] = \sum_{k=0}^{\infty} b_k(X)x^k$.*

Up to now we have denoted the space of long knots (modulo immersions) by $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$. For the sake of simplicity, we will denote it here by $\mathcal{K}$. Let $\mathcal{K}^{\times m}$ denote the space of m copies of long knots. Recall also the notation $\overline{\mathcal{L}}_m^d$ for the space of long links (modulo immersions) of m strands in $\mathbb{R}^d$. It is clear that $\mathcal{K}^{\times m}$ is a subspace of $\overline{\mathcal{L}}_m^d$.

Definition 5.6.2. *The pair $(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m})$ is called the* space of long links modulo m copies of long knots*.*

In [24], Komawila and Lambrechts studied the Euler series of the E_1 page of the H^*BKSS associated to the Munson-Volić cosimplicial model (we have seen it in Section 5.4, and we have denoted it by $\mathcal{L}_m^{d\bullet}$) for the space of long links, and they obtained the following theorem and corollary. Recall that the Euler series associated to a bigraded vector space $V = \{V_{p,q}\}_{p,q\geq 0}$ is defined by

$$\chi(V)[x] = \sum_{q=0}^{\infty}\left(\sum_{p=0}^{\infty}(-1)^p \dim(V_{p,q})\right)x^q.$$

Theorem 5.6.3. [24, Theorem 5.1] *For $d \geq 4$ the Euler series $\chi(E_1)[x]$ of the E_1 page of the H^*BKSS associated to $\mathcal{L}_m^{d\bullet}$ is given by*

$$\chi(E_1)[x] = \frac{1}{(1-x^{d-1})(1-2x^{d-1})\cdots(1-mx^{d-1})}. \tag{5.6.1}$$

The following corollary gives the Euler series of the pair $(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m})$. Before stating it, we recall that the pair $(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m})$ admits a cosimplicial model $(\mathcal{L}_m^{d\bullet}, (\mathcal{K}_d^\bullet)^{\times m})$. The second component of that cosimplicial model is just the product $(\mathcal{K}_d^\bullet)^{\times m}$ of m copies of the Sinha cosimplicial model $\mathcal{K}_d^\bullet$.

Corollary 5.6.4. [24] *For $d \geq 4$ the Euler series of the E_2 page of the H^*BKSS associated to the pair $(\mathcal{L}_m^{d\bullet}, (\mathcal{K}_d^\bullet)^{\times m})$ is given by*

$$\chi(E_2)[x] = \frac{1}{(1-x^{d-1})(1-2x^{d-1})\cdots(1-mx^{d-1})} - \frac{1}{(1-x^{d-1})^m}. \tag{5.6.2}$$

Proof. The proof comes from Theorem 5.6.3 and the fact that the retraction (up to homotopy) $\overline{\mathcal{L}}_m^d \longrightarrow \mathcal{K}^{\times m}$ (see [24, Section 2.1] for an explicit definition of that retraction) holds at the level of cosimplicial models so that we have the following isomorphism of spectral sequences

$$\{E_r((\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m}))\}_{r\geq 0} \cong \frac{\{E_r(\overline{\mathcal{L}}_m^d)\}_{r\geq 0}}{\{E_r(\mathcal{K}^{\times m})\}_{r\geq 0}}. \tag{5.6.3}$$

□

From Corollary 5.6.4 and our collapsing Theorem 5.1.6, we have the growth of the Betti numbers of the pair $(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m})$.

Proposition 5.6.5. *For $d > 5$ the Betti numbers of the pair $(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m})$ have an exponential growth.*

Proof. By (5.6.3) and Theorem 5.1.6, the H^*BKSS of the pair $(\mathcal{L}_m^{d\bullet}, (\mathcal{K}_d^\bullet)^{\times m})$ collapses at the E^2 page. Moreover the coefficients of (5.6.2) have an exponential growth of rate $m^{\frac{1}{d-1}} > 1$, and by [24, Proposition 4.5] the Betti numbers of the pair $(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m})$ have the same growth. □

One can also see Proposition 5.6.5 as a consequence of a theorem of Turchin (see [53, Theorem 17.1]), which states that the Betti numbers of the space $\mathcal{K}$ of long knots grow at least exponentially. Notice first that the concatenation operation endows $\overline{\mathcal{L}}_m^d$ and $\mathcal{K}^{\times m}$ with a structure of H-space. Let $\mathbf{1} \in \mathcal{K}^{\times m}$ denote the unit, and consider the diagram

$$\begin{tikzcd} F \arrow[r, "i"] & \overline{\mathcal{L}}_m^d \arrow[r, "\rho"] & \mathcal{K}^{\times m} \\ F \arrow[u, "id"] \arrow[r, "g"] & F \times \mathcal{K}^{\times m} \arrow[u, "\psi"] \arrow[r, "f"] & \mathcal{K}^{\times m}, \arrow[u, "id"'] \end{tikzcd} \tag{5.6.4}$$

where

- F is the fiber of ρ over the unit $\mathbf{1}$, id is the identity map,
- the map ψ is defined by $\psi(x,y) = i(x) \times s(y)$, where $s\colon \mathcal{K}^{\times m} \longrightarrow \overline{\mathcal{L}}_m^d$ is a section of ρ,
- the maps g and f are defined by $g(x) = (x, \mathbf{1})$ and $f(x,y) = y$,
- the map ρ is the one constructed in [24, Section 2].

It is clear that the left square of (5.6.4) commutes. The right square also commutes because of the following

$$\begin{aligned}\rho(\psi(x,y)) &= \rho(i(x) \times s(y)) \\ &= \rho(i(x)) \times \rho(s(y)) \text{ because } \rho \text{ is a morphism of H-spaces} \\ &= \mathbf{1} \times y \text{ because } s \text{ is a section of } \rho \\ &= f(x,y).\end{aligned}$$

This implies that the triple (id, ψ, id) is a morphism of fibrations, and therefore the space $\overline{\mathcal{L}}_m^d$ is homeomorphic to the product $F \times \mathcal{K}^{\times m}$. We thus have the following inequality

$$\dim(H_*(\overline{\mathcal{L}}_m^d)) > \dim(H_*(\mathcal{K}^{\times m})).$$

Since $\dim(H_*(\mathcal{K}^{\times m}))$ grows exponentially (because, by Theorem 17.1 in [53], the dimension $\dim(H_*(\mathcal{K}))$ grows exponentially), the Proposition 5.6.5 follows.

We will make a remark on the Turchin approach and on our approach in proving Proposition 5.6.5. In the proof of Proposition 5.6.5 we have seen that the Betti numbers of the pair $(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m})$ have an exponential growth of rate $m^{\frac{1}{d-1}}$. This implies the following corollary.

Corollary 5.6.6. *For $d > 5$, the radius of convergence of the Poncaré series for the pair $(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m})$ is less than or equal to $(\frac{1}{m})^{\frac{1}{d-1}}$, and therefore tends to 0 as m goes to ∞.*

Let us denote by $RC(X)$ the radius of convergence of the Poincaré series for a space X. Specially for the space of long knots, we will denote it by R.

Remark 5.6.7. *As a consequence of Theorem 5.1.10 we have the inequality $RC(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m}) \leq (\frac{1}{\sqrt{2}})^{\frac{1}{d-1}}$. Our approach gives a better upper bound of this radius because of Corollary 5.6.6.*

We end this section with a conjecture (we believe in that conjecture) and a theorem.

Conjecture 5.6.8. *The radius of convergence of the Poincaré series of the space of long knots (modulo immersions) is greater than 0. That is, $R > 0$.*

The Corollary 5.6.6 tells us that the radius of convergence of the Poincaré series for $(\overline{\mathcal{L}}_m^d, \mathcal{K}^{\times m})$ is less than or equal to $(\frac{1}{m})^{\frac{1}{d-1}}$, but does not tell us that it is less than the one R of the space of long knots. We therefore have the following theorem.

Theorem 5.6.9. *If Conjecture 5.6.8 is true, then for $d > 5$ and for $m > \frac{1}{R^{d-1}}$ the radius of convergence of the Poincaré series for the pair $(\overline{\mathcal{L}}^d_m, \mathcal{K}^{\times m})$ is less than R. That is, $RC(\overline{\mathcal{L}}^d_m, \mathcal{K}^{\times m}) < R$.*

Proof. The proof comes immediately from Corollary 5.6.6 and the hypothesis $m > \frac{1}{R^{d-1}}$. $\square$

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