Homework week 3 solutions

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Lagrange Multipliers

0.1 Gradient

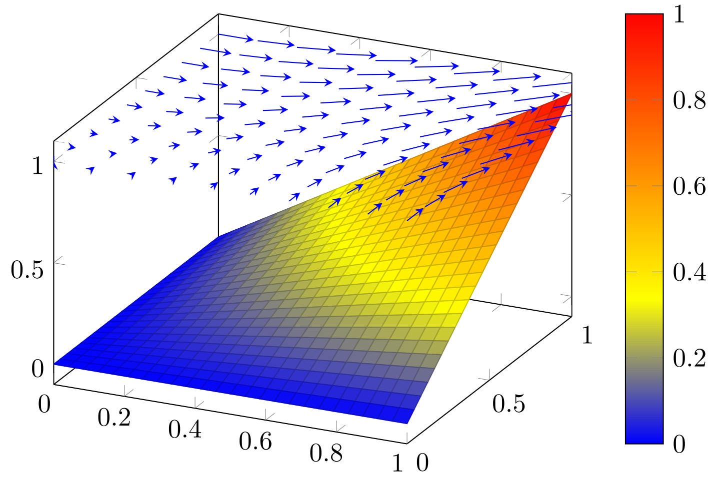
Definition 1. Gradient. Let f be differentiable at (x, y). The **gradient** of $f(x, y)$ is denoted by $\nabla f(x, y)$ and defined by:

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$$

It can be noticed that $\nabla f(x, y)$ is a vector. The symbol ∇ is called *nabla*.

Example 0.1. Find the gradient of $f(x, y) = xy$.
Let's find the first derivatives:

$$f_x(x,y) = y; \qquad f_y(x,y) = x$$



So, $\nabla f(x, y) = \langle y, x\rangle$

Example 0.2. Find the gradient of $f(x, y) = x^2 + 2xy + 3y^2$.
Let's find the first derivatives:

$$f_x(x,y) = 2x + y; \qquad f_y(x,y) = 2x + 6y$$

So, $\nabla f(x, y) = \langle 2x + y, 2x + 6y\rangle$

Example 0.3. Find the gradient of $f(x, y) = \ln(xy)$.

Let's find the first derivatives using $(\ln u)' = \dfrac{u'}{u}$

$$f_x(x,y) = \frac{y}{xy} = \frac{1}{x}; \qquad f_y(x,y) = \frac{x}{xy} = \frac{1}{y}$$

So, $\nabla f(x,y) = \left\langle \frac{1}{x}, \frac{1}{y} \right\rangle$

Example 0.4. Find the gradient of $f(x,y) = x^2 y e^{xy}$.
Let's find the first derivatives using $(u \cdot v)' = u' \cdot v + u \cdot v'$

$$f_x(x,y) = \frac{\partial(x^2y)}{\partial x} \cdot e^{xy} + x^2y \cdot \frac{\partial(e^{xy})}{\partial x} = 2xye^{xy} + x^2y^2e^{xy} = (2xy + x^2y^2)e^{xy}$$
$$f_y(x,y) = \frac{\partial(x^2y)}{\partial y} \cdot e^{xy} + x^2y \cdot \frac{\partial(e^{xy})}{\partial y} = x^2e^{xy} + x^3ye^{xy} = (x^2 + x^3y)e^{xy}$$

So, $\nabla f(x,y) = \left\langle (2xy + x^2y^2)e^{xy}, (x^2 + x^3y)e^{xy} \right\rangle$

0.2 Lagrange Multipliers

In the previous section we optimized (i.e. found the absolute extrema) a function on a region that contained its boundary. In this section we are going to take a look at another way of optimizing a function subject to given constraint(s).

Definition 2. Objective function. It is the function $f(x,y)$ that we wish to optimize.

Definition 3. Constraint. It is a curve $\mathcal{C}$ in the xy-plane on which we wish to find the min/max of the function $f(x,y)$. It is defined by $g(x,y) = 0$.

Method

Let $f(x,y)$ be the objective function and $g(x,y)$ the constraint with $\nabla g(x,y) \neq 0$ on the curve $g(x,y) = 0$. The following steps give the min/max of the function $f(x,y)$ subjected to the constraint $g(x,y) = 0$:

1. Find the values of x, y and λ that satisfy the following system of 2 equations:

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$g(x,y) = 0$$

   λ is called *Lagrange Multiplier.*

2. From Step 1, the largest (resp. smallest) value gives the maximum (resp. minimum) of the function $f(x,y)$ at the point (x,y) subjected to the constraint $g(x,y) = 0$.

Example 0.5. Use the Lagrange Multipliers to find the minimum and maximum values of $f(x,y) = x^2 + y^2 - 2x + 2y + 5$ on the curve $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 4\}$.

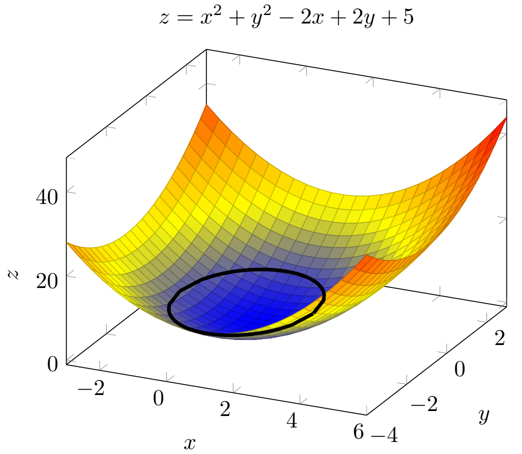
Solution: Here $g(x,y) = x^2 + y^2 - 4$. On the curve $\mathcal{C}$ the value of f is decreasing after the point P. The point P can be characterised by the point where the gradients of f and g are parallel (orthogonal). It is where the value of f on the curve $\mathcal{C}$ is the maximum.

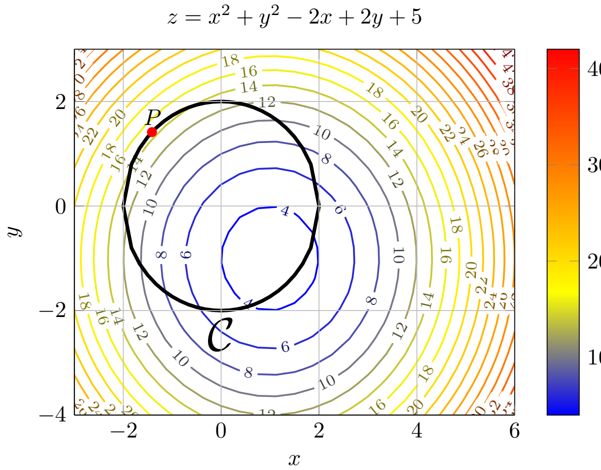
Method:

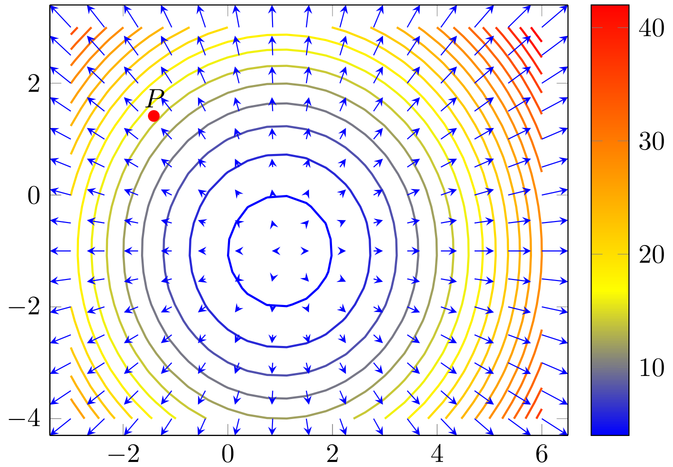
1. Find the value of x, y and λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$g(x,y) = 0$$

2. From step 1 chose the set of (x, y, λ) that gives the largest and the smallest value of $f(x,y)$







Here we have:

$$\nabla f = \langle 2x - 2, 2y + 2 \rangle$$

$$\nabla g = \langle 2x, 2y \rangle$$

We now have the system of two equations as follows:

$$2x - 2 = \lambda 2x$$

$$2y + 2 = \lambda 2y$$

After simplification, we have:

$$x - 1 = \lambda x \tag{1}$$

$$y + 1 = \lambda y \tag{2}$$

Here we will try to eliminate λ. From Eq. (1) and Eq. (2), we can write:

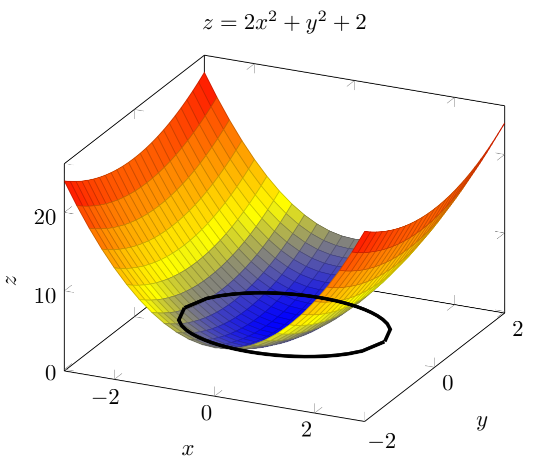
$$\lambda = \frac{x-1}{x}$$

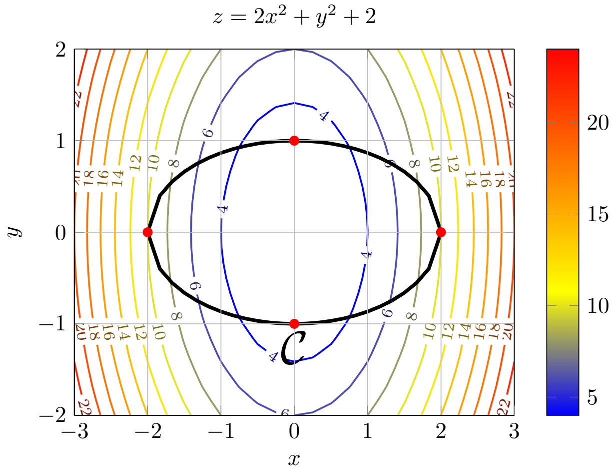
$$\lambda = \frac{y+1}{y}$$

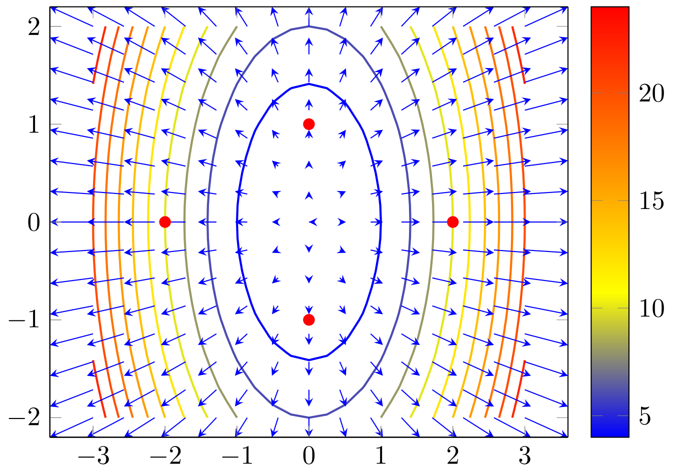
This gives us $\dfrac{x-1}{x} = \dfrac{y+1}{y} \Longrightarrow xy + x = xy - y \Longrightarrow x = -y$. Since $g(x, y) = x^2 + y^2 - 4 = 0$, we have $x^2 + (-x)^2 = 4 \Longrightarrow x = \sqrt{2} \Longrightarrow y = -\sqrt{2}$ or $x = -\sqrt{2} \Longrightarrow y = \sqrt{2}$

$\boxed{\text{So we have } f(\sqrt{2}, -\sqrt{2}) = 9 - 4\sqrt{2} \text{ and } f(-\sqrt{2}, \sqrt{2}) = 9 + 4\sqrt{2}}$

Example 0.6. Use the Lagrange Multipliers to find the minimum and maximum values of $f(x,y) = 2x^2+y^2+2$ on the curve $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 | x^2 + 4y^2 = 4\}$.







Here $\mathcal{C}$ is an ellipse: $g(x,y) = \dfrac{x^2}{4} + y^2 - 1 = 0$. $\nabla f = \lambda \nabla g$ with $g(x,y) = 0$ give:

$$2x = \lambda x \tag{3}$$

$$y = 4\lambda y \tag{4}$$

$$x^2 + 4y^2 - 4 = 0 \tag{5}$$

Then from Eq. (3) and Eq. (4) we have:

$$x(\lambda - 2) = 0 \tag{6}$$

$$y(4\lambda - 1) = 0 \tag{7}$$

Eq. (6) gives $x = 0$ or $\lambda = 2$.
If $\lambda = 2$, we have (from Eq. (6)): $y(8-1) = 0 \Longrightarrow y = 0$. So Eq. (5) gives: $x^2 = 4 \Longrightarrow x = \pm 2$. We have two critical points $(2,0)$ and $(-2,0)$.
If $x = 0$, then from Eq. (5), we have $y^2 = 1 \Longrightarrow y = \pm 1$. So we have two critical points $(0,-1)$ and $(0,-1)$.

Now we have to compute the value of $f(x, y)$ at these critical points. We have:

$$\begin{aligned} f(2,0) &= 10 \\ f(-2,0) &= 10 \\ f(0,1) &= 3 \\ f(0,-1) &= 3 \end{aligned}$$