Integral Calculus

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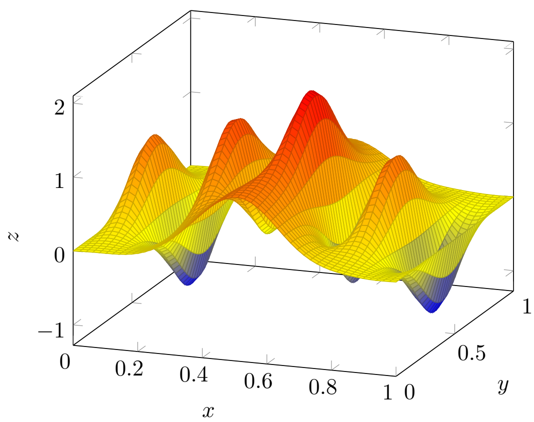
Minimum/Maximum Problems (sec. 12.8)

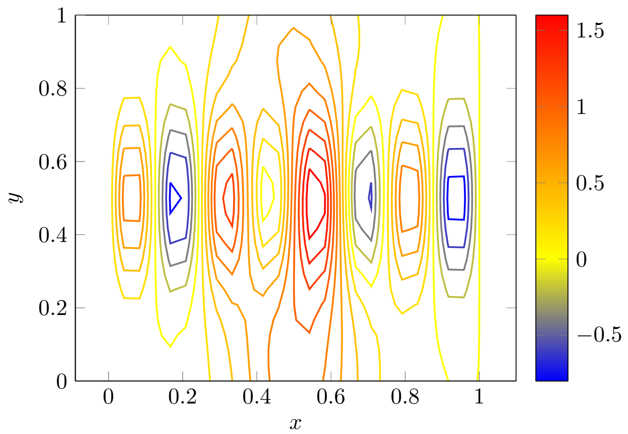
A function $f(x, y)$ has a **local maximum value** at a point (a, b) if $f(x, y) \leqslant f(a, b)$ for **all** points (x, y) in some open disk centered at data point.

A function $f(x, y)$ has a **local minimum value** at a point (a, b) if $f(x, y) \geqslant f(a, b)$ for **all** points (x, y) in some open disk centered at data point.

These points are also called **local extreme values** or **local extrema**.

Theorem 1. If a function $f(x, y)$ has a local minimum (or a minimum) value at (a, b) and the partial derivatives $f_x \left(\dfrac{\partial f}{\partial x}\right)$ and $f_y \left(\dfrac{\partial f}{\partial y}\right)$ exist (not ∞) at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.





Definition 1. An interior point (a, b) in D_f is a **critical point** of the function $f(x, y)$ if either:

- $f_x(a, b) = f_y(a, b) = 0$
- at least one of f_x and f_y does not exist at (a, b)

Strategy

1. Find the first derivative f_x and f_y
2. Find the point (a, b) for which $f_x(x, y) = f_y(x, y) = 0$

Related Exercises sec. 12.8 9–18

Example 0.1. $f(x,y) = 5xy(x-4)(y+6) = 5(y^2+6y)(x^2-4x)$. Find critical points.
First, let's compute the first derivative.

$$f_x = 5(y^2+6y)(2x-4) \tag{1}$$
$$f_y = 5(2y+6)(x^2-4x) \tag{2}$$

Then we have,

$$f_x = 5(y^2+6y)(2x-4) = 5y(y+6)(2x-4) = 0 \tag{3}$$
$$f_y = 5(2y+6)(x^2-4x) = 5x(2y+6)(x-4) = 0 \tag{4}$$

From Eq. (3) we have:

$$y = 0 \quad \text{or} \quad y+6=0 \quad \text{or} \quad 2x-4=0 \Longrightarrow y=0 \quad \text{or} \quad y=-6 \quad \text{or} \quad x=2$$

Now, we *plug* $x=2$ into Eq. (4):

$$5(2)(2y+6)(2-4) = 0 \Longrightarrow -20(2y+6) = 0 \Longrightarrow y = -3$$

So, $(2,-3)$ is a critical point.

Then, we *plug* $y=0$ into Eq. (4):

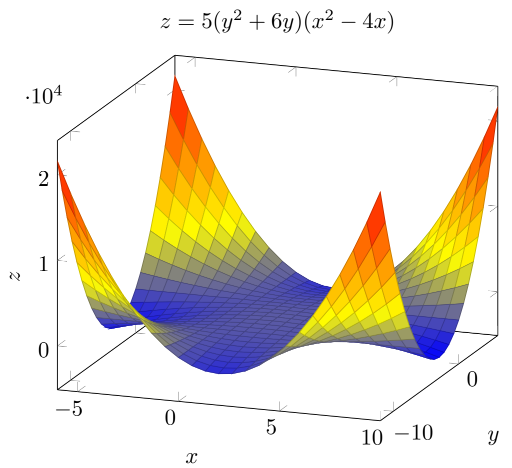
$$5x(2(0)+6)(x-4) = 0 \Longrightarrow 30x(x-4) = 0 \Longrightarrow x=0 \quad \text{or} \quad x=4$$

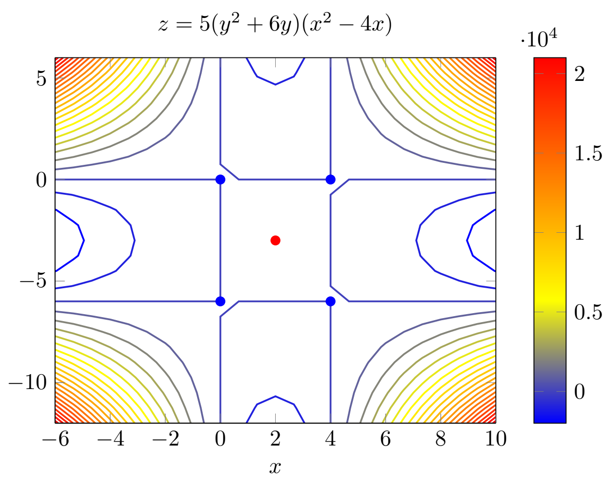
So $(0,0)$ and $(4,0)$ are critical points.

Finally, we *plug* $y=-6$ into Eq. (4):

$$5x(2(-6)+6)(x-4) = 0 \Longrightarrow -30x(x-4) = 0 \Longrightarrow x=0 \quad \text{or} \quad x=4$$

So $(0,-6)$ and $(4,-6)$ are critical points.





Example 0.2. $f(x,y) = x^2 + \frac{1}{3}y^3 + 2xy - 3y$. Find the critical points.
First, let's compute the first derivative.

$$f_x = 2x + 0 + 2y + 0 = 2x + 2y \tag{5}$$
$$f_y = 0 + y^2 + 2x - 3 = y^2 + 2x - 3 \tag{6}$$

Then, we have:

$$f_x = 2x + 2y = 0 \tag{7}$$

$$f_y = y^2 + 2x - 3 = 0 \tag{8}$$

First, from EQ. (7), we have $x = -y$. Then we have to *plug* it into EQ. (8) to get:

$$y^2 + 2(-y) - 3 = 0 \Longrightarrow y^2 - 2y - 3 = 0$$

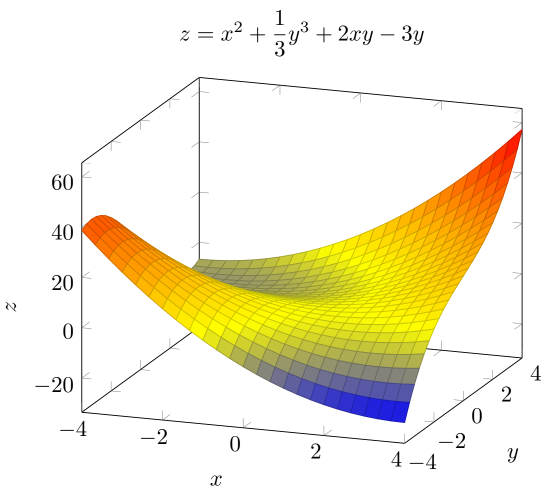
Now we have a quadratic equation. So the roots are expressed as:

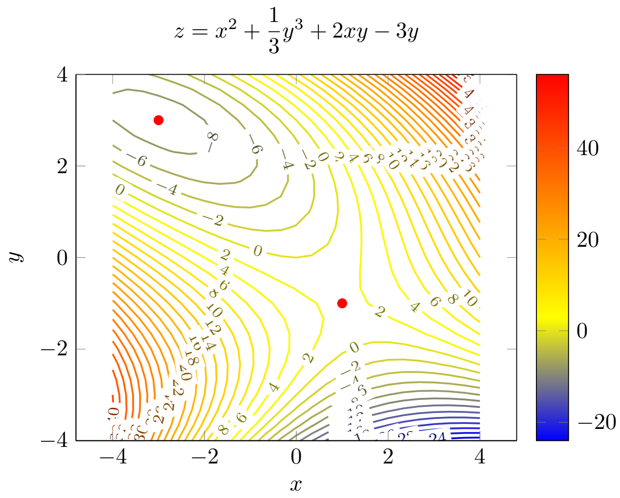
$$y = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)(1)}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2}$$

$$y = \frac{6}{2} = 3 \quad \text{and} \quad y = -1$$

Since $x = -y$, if $y = -1$ then $x = 1$ and if $y = 3$ then $x = -3$.

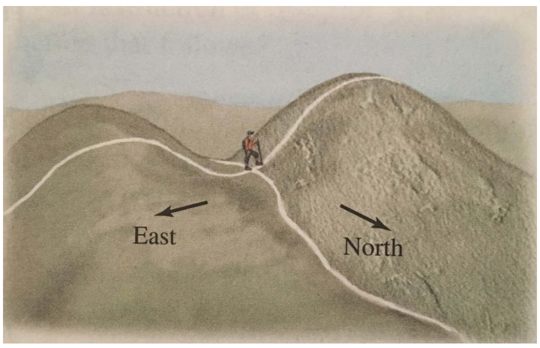
So $(1, -1)$ and $(-3, 3)$ are critical points.





0.1 Saddle Point

Definition 2. A function $f(x, y)$ has a **saddle point** at a critical point (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$. It means that if (a, b) is a critical point of f and f has a saddle point at (a, b), they form the point $(a, b, f(a, b))$. It is possible to walk uphill in some directions and downhill in other directions.

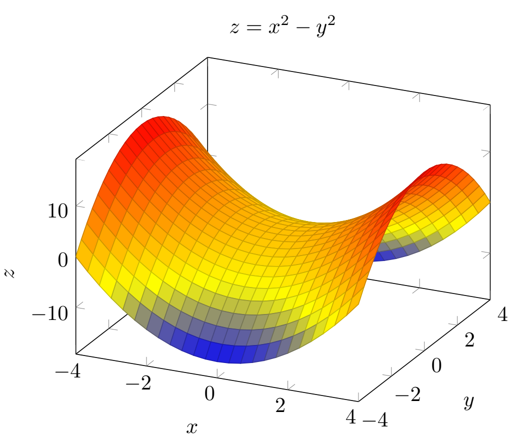


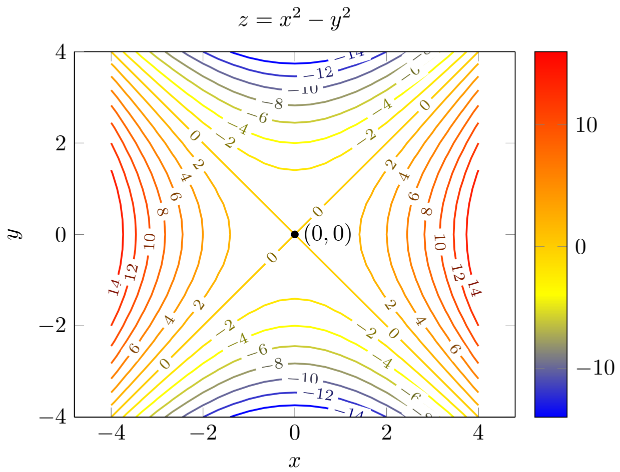
Example 0.3. The hyperbolic paraboloid $f(x, y) = x^2 - y^2$ has a saddle point at $(0, 0)$

0.2 Analysing critical points

Theorem 2. Second Derivative Test. Suppose that the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$.

Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$.





1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b)
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b)
3. If $D(a, b) < 0$, then f has a saddle point at (a, b)
4. If $D(a, b) = 0$, then the test is inconclusive

Related Exercises sec. 12.8 19–34

Example 0.4. Analysing critical points. Use the Second Derivative Test to classify the critical points of $f(x, y) = xy(x - 2)(y + 3)$. First, we compute the first partial derivatives.

$$
\begin{aligned}
f_x &= \frac{\partial}{\partial x}[xy(x-2)] \cdot (y+3) + xy(x-2) \cdot \frac{\partial}{\partial x}(y+3) \\
f_x &= \left[\frac{\partial}{\partial x}(xy) \cdot (x-2) + xy\frac{\partial}{\partial x}(x-2)\right](y+3) \\
f_x &= [y(x-2) + xy](y+3) \\
f_x &= 2y(x-1)(y+3) \\
f_y &= \frac{\partial}{\partial y}[xy(x-2)] \cdot (y+3) + xy(x-2) \cdot \frac{\partial}{\partial y}(y+3) \\
f_y &= \left[\frac{\partial}{\partial y}(xy) \cdot (x-2) + xy\frac{\partial}{\partial y}(x-2)\right] \cdot (y+3) + xy(x-2) \\
f_y &= x(x-2)(y+3) + xy(x-2) \\
f_y &= (x-2)[x(y+3) + xy] \\
f_y &= (x-2)(2xy + 3x) \\
f_y &= x(x-2)(2y+3)
\end{aligned}
$$

And we find the points corresponding to $f_x = 0$ and $f_y = 0$.

$$f_x = 2y(x-1)(y+3) = 0 \tag{9}$$
$$f_y = x(x-2)(2y+3) = 0 \tag{10}$$

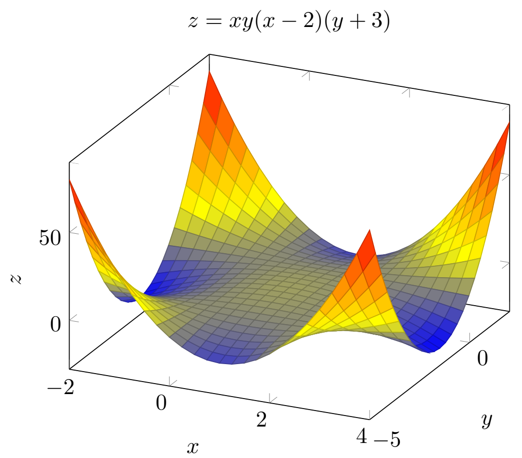
We have $y = 0$ or $x = 1$ or $y = -3$. We consider each of these cases.

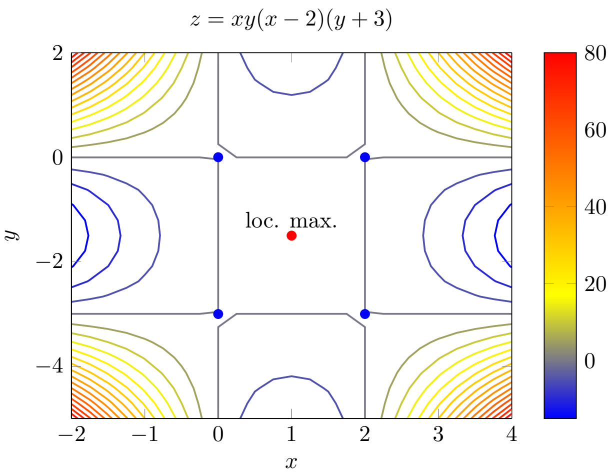
- Substituting $y = 0$, Eq. (10) gives the equation $3x(x-2) = 0$, which has solutions $x = 0$ and $x = 2$. So $(0,0)$ and $(2,0)$ are critical points.
- Substituting $x = 1$, Eq. (10) gives the equation $-(2y+3) = 0$, which has solutions $y = -3/2$. So $(1,-3/2)$ is a critical point.
- Substituting $y = -3$, Eq. (10) gives the equation $-3x(x-2) = 0$, which has roots $x = 0$ and $x = 2$. So $(0,-3)$ and $(2,-3)$ are critical points.

We just found that we have 5 critical points: $(0,0)$, $(2,0)$, $(1,-3/2)$, $(0,-3)$ and $(2,-3)$. Now we should calculate $D(x,y)$ but before that we need to find the second partial derivatives which are:

$$f_{xx} = 2y(y+3); \qquad f_{xy} = 2(2y+3)(x-1); \qquad f_{yy} = 2x(x-2)$$

Points	f_{xx}	f_{yy}	$(f_{xy})^2$	$D(x,y)$	Conclusion
$(0,0)$	0	0	36	-36	Saddle point
$(2,0)$	0	0	36	-36	Saddle point
$(1,-3/2)$	$-9/2$	2	0	9	Local maximum
$(0,-3)$	0	0	36	-36	Saddle point
$(2,-3)$	0	0	36	-36	Saddle point





0.3 Optimisation

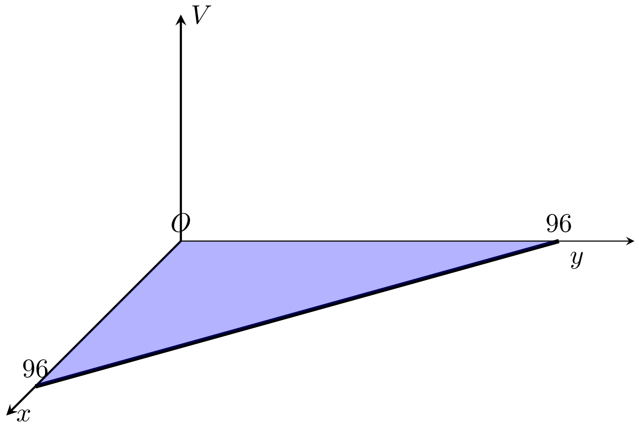
Example 0.5. Shipping regulations. A shipping company handles rectangular boxes provided the sum of the length, width and height of the box does not exceed 96. Find the dimension of the box that meets this condition and has the largest volume.

Solution. Let x, y, z be the dimensions of the box and $V = xyz$ it volume. The box maximum volume satisfies the $x + y + z = 96$, which is used to eliminate any one of the variables from the volume function. Noting that $z = 96 - x - y$, the volume function becomes:

$$V(x, y) = xy(96 - x - y)$$

Notice that dimensions are non-negative!!! So we have the condition:

$$96 - x - y \geqslant 0 \qquad \Longrightarrow \qquad x + y \leqslant 96$$



The solution is met if (x, y) lies in the triangle bounded by $x = 0$, $y = 0$ and $x + y = 96$. Here the domain is the triangle and $V = 0$ on its boundary.
The goal is to find the critical points of V that satisfy:

$$V_x = 96y - 2xy - y^2 = y(96 - 2x - y) = 0 \qquad (11)$$

$$V_y = 96x - 2xy - x^2 = x(96 - 2y - x) = 0 \qquad (12)$$

Eq. (11) gives $y = 0$ or $y = 96 - 2x$. When we *plug* these values into Eq. (12), we have:

$$y = 0 \Longrightarrow \qquad x = 96 - 2(0) = 96$$

$$y = 96 - 2x \Longrightarrow \quad x = 96 - 2(96 - 2x) \Longrightarrow x = 32, \quad \text{so } y = 96 - 2(32) = 32$$

So $(96, 0)$ and $(32, 32)$ are critical points.

Eq. (12) gives $x = 0$ or $x = 96 - 2y$. When we *plug* these values into Eq. (11), we have:

$$x = 0 \Longrightarrow y = 0$$

$$x = 96 - 2y \Longrightarrow \quad y = 96 - 2(96 - 2y) \Longrightarrow y = 32, \quad \text{so } x = 96 - 2(32) = 32$$

So $(0, 0)$ and $(0, 96)$ are critical points.

In summary, we have 4 critical points (0,0), (0,96), (96,0) and (32,32). We can see that the first 3 solutions lie on the boundary of the domain, where $V = 0$. Therefore, the remaining critical point is $(32, 32)$.

Now, we apply the Second Derivative Test to check the type of critical point at $(32, 32)$.

$$D(x, y) = V_{xx}(x, y)V_{yy}(x, y) - (V_{xy}(x, y))^2$$

$$\text{where } V_{xx} = -2y; \quad V_{xy} = 96 - 2x - 2y; \quad V_{yy} = -2y$$

$$\text{So} \quad D(x, y) = 4xy - (96 - 2x - 2y)^2$$

Substituting the value of (x, y) by $(32, 32)$, we have $D(32, 32) = 3072 > 0$.

Since $V_{xx}(32, 32) = -2(32) = -64 < 0$, we have a local maximum at $(32, 32)$. The dimensions of the box with maximum volume are $x = 32$, $y = 32$ and $z = 96 - 32 - 32 = 32$ (it's a cube). Its volume is $32 \times 32 \times 32 = 32,768$ which is the maximum volume of the domain.