Integral Calculus

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1 Substitution Rule

The substitution rule helps us to compute an Indefinite and definite integrals.

Example 1.1. Evaluate the following indefinite integral:

$$\int 6x^2(2x^3-5)^{10}dx$$

Let's choose $u = 2x^3 - 5$. When we differentiate u, we have $\frac{du}{dx} = 6x^2$. Now, by substituting in the previous expression, we have:

$$\int 6x^2(2x^3-5)^{10}dx = \int \frac{du}{dx}u^{10}dx = \int u^{10}du = \frac{u^{11}}{11} + c$$

The expression become easier to resolve. The after putting back the expression of u, we have:

$$\int 6x^2(2x^3-5)^{10}dx = \frac{(2x^3-5)^{11}}{11} + c$$

Example 1.2. Evaluate the following indefinite integral:

$$\int_0^3 \frac{2x}{\sqrt{x^2+16}}dx$$

Let's choose $u = x^2+16$. When we differentiate u, we have $\frac{du}{dx} = 2x$. The upper limit becomes $u = 3^2+16 = 15$ and the lower limit becomes $u = 0^2 + 16 = 16$ Now, by substituting in the previous expression, we have:

$$\int_0^3 \frac{2x}{\sqrt{x^2+16}}dx = \int_{16}^{25} \frac{\frac{du}{dx}}{\sqrt{u}}dx = \int_{16}^{25} \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_{16}^{25} = 2$$

Indefinite Integral

Let $u = g(x)$, where $g'(x)$ is continuous on an interval, and let $f(x)$ continuous on that interval. On that interval

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du$$

Definite Integral

Let $u = g(x)$, where $g'(x)$ is continuous on $[a, b]$, and let $f(x)$ continuous on the range of $g(x)$, then:

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Procedure

Indefinite Integral

1. Find the best option for substitution $u = g(x)$
2. Differentiation: $du = g'(x)dx$
3. Substitute $u = g(x)$ and $du = g'(x)dx$ in the integral
4. Evaluate the new integral with respect to u
5. Write the result as a function of x using $u = g(x)$. Do not forget to add the constant C

Definite Integral

1. Find the best option for substitution $u = g(x)$
2. Differentiation: $du = g'(x)dx$
3. Find the new upper and lower bounds $g(a)$ and $g(b)$
4. Evaluate the new integral
5. Write the result, which is a **real number**

Example 1.3. Evaluate the following indefinite integral:

$$\int e^{-3x}dx$$

1. Let's choose $u = -3x$
2. Differentiation $du = -3dx$
3. Substitution
$$\int e^{-3x}dx = \int e^u \frac{du}{-3} = -\frac{1}{3}\int e^u du$$
4. Evaluation
$$-\frac{1}{3}\int e^u du = -\frac{e^u}{3} + c$$
5. Substitution
$$\int e^{-3x}dx = -\frac{e^u}{3} + c = -\frac{e^{-3x}}{3} + c$$

Example 1.4. Evaluate the following indefinite integral:

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos x dx$$

1. Let's choose $u = \sin x$
2. Differentiation $du = \cos x dx$
3. New upper and lower bounds and substitution: $g(0) = \sin(0) = 0$ and $g(\pi/2) = \sin(\pi/2) = 1$
$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos x dx = \int_0^1 u^4 du = \frac{u^5}{5}\bigg|_0^1$$
4. Evaluation
$$\int_0^1 u^4 du = \frac{u^5}{5}\bigg|_0^1 = \frac{1}{5} - 0$$
5. Write the result
$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos x dx = \frac{1}{5}$$

Example 1.5. Evaluate the following indefinite integral (Integral of $\cos^2 x$ or $\sin^2 x$):

Hint: use the formula $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

$$\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta + \int_0^{\frac{\pi}{2}} \frac{\cos(2\theta)}{2} d\theta$$

1. Let's choose $u = 2\theta$
2. Differentiation $du = 2d\theta$
3. New upper and lower bounds and substitution: $g(0) = 2(0) = 0$ and $g(\pi/2) = 2(\pi/2) = \pi$

$$\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta + \int_0^{\pi} \frac{\cos(u)}{2} du$$

4. Evaluation

$$\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \frac{1}{2}\theta\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\sin(u)\Big|_0^{\pi} = \frac{1}{2}\cdot\frac{\pi}{2} + 0$$

5. Write the result

$$\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \frac{\pi}{4}$$