Integral Calculus

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Planes in $\mathbb{R}^3$

0.1 Recall of dot product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. The dot product is defined as:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 + v_2 + u_3 v_3$$

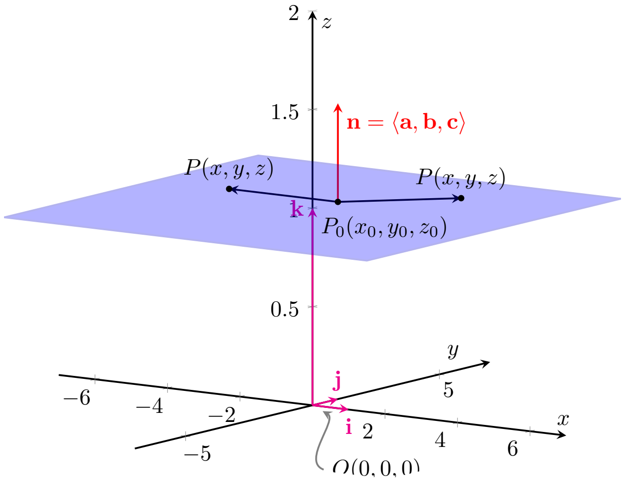
Two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are parallel if:

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3} = c$$

Two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

0.2 Plane passing through a point and perpendicular to a vector

Given a fixed point $P_0(x_0, y_0, z_0)$ and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ is called a **plane**. (cf. p.858)



If a point $P(x, y, z)$ is on the plane, $\overrightarrow{P_0P}$ must be orthogonal to the normal vector $\mathbf{n} = \langle a, b, c \rangle$. On the one hand, any point $P(x, y, z)$ on the plane, we know that $\overrightarrow{P_0P}$ must be orthogonal to $\mathbf{n} = \langle a, b, c \rangle$. On the other hand, we know that $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$. Therefore, by definition $\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$ ($\overrightarrow{P_0P}$ orthogonal to $\mathbf{n}$). The result of the dot product is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

After a simplification, we have:

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$
$$ax + by + cz = ax_0 + by_0 + cz_0$$

where $d = ax_0 + by_0 + cz_0$. Therefore,

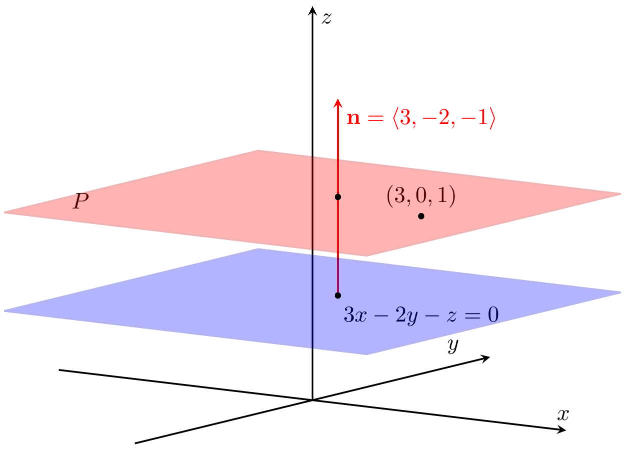
$$ax + by + cz = d$$

Related Exercises sec. 12.1 (11–16)

Example 0.1. Find the equation of the plane passing through $P_0(-1, 1, 1)$ with a normal vector $\mathbf{n} = \langle 2, -1, 4\rangle$. We have P_0 and $\mathbf{n}$ so we just need to replace the variables in the general formula to get the equation.
Here $P_0(x_0, y_0, z_0) = P_0(-1, 1, 1)$ and $\mathbf{n} = \langle a, b, c\rangle = \langle 2, -1, 4\rangle$.
So,

$$2(x - (-1)) + (-1)(y - 1) + 4(z - 1) = 0$$
$$2(x + 1) - 1(y - 1) + 4(z - 1) = 0$$
$$2x + 2 - y + 1 + 4z - 4 = 0$$
$$2x - y + 4z = 1$$

Example 0.2. Find an equation of the plane P passing through the point (3,0,1) and is parallel to the plane $3x - 2y - z = 0$. We know that the vector $\mathbf{n} = \langle 3, -2, -1\rangle$ is the normal to the plane $3x - 2y - z = 0$. Now,

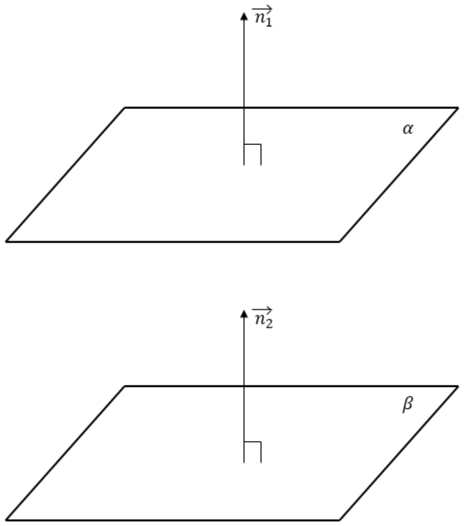


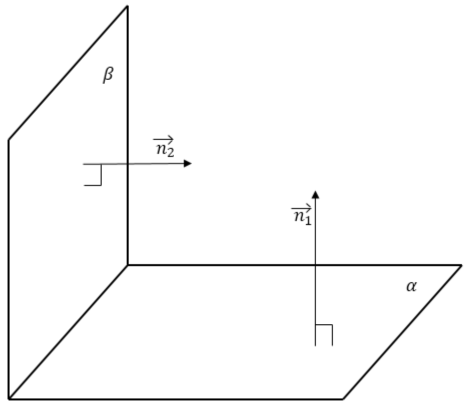
because the planes P and $3x - 2y - z = 0$ are parallel, the vector $\mathbf{n} = \langle 3, -2, -1\rangle$ is also orthogonal to P and it is also a normal vector to P so,

$$P : 3(x - 3) - 2(y - 0) - 1(z - 1) = 0$$
$$\boxed{P : 3x - 2y - z = 8}$$

0.3 Orthogonal and parallel planes

Two distinct **planes** (Q and R) are **parallel** if their respective **normal vectors** are **parallel**. It means that the normal vectors are scalar multiple of each other ($\mathbf{n}_Q = c\mathbf{n}_R$).
Two planes (Q and R) are **orthogonal** if their respective **normal vectors** are **orthogonal**. It means that the dot product of the normal vectors are zero ($\mathbf{n}_Q \cdot \mathbf{n}_R = 0$). *Related Exercises sec. 12.1 (25–30)*





Note: How to find out if two planes P_1 and P_2 are orthogonal or parallel?

- Find the normal vectors $\mathbf{n}_1 = \langle a_1, b_1, c_1 \rangle$ and $\mathbf{n}_2 = \langle a_2, b_2, c_2 \rangle$
- If $\mathbf{n}_1$ and $\mathbf{n}_2$ parallel $\Longrightarrow$ P_1 and P_2 are parallel. It means:
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = c$$
- If $\mathbf{n}_1$ and $\mathbf{n}_2$ orthogonal $\Longrightarrow$ P_1 and P_2 are orthogonal. It means:
$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$$

Example 0.3. Which of the following distinct planes are parallel and which are orthogonal?

$$Q : 2x - 3y + 6z = 12 \qquad R : -x + \frac{3}{2}y - 3z = 14$$
$$S : 6x + 8y + 2z = 1 \qquad T : -9x - 12y - 3z = 7$$

First, let's find the normal vectors. They are:

$$Q : \mathbf{2}x \mathbf{-3}y + \mathbf{6}z = 12 \Rightarrow \mathbf{n}_Q = \langle 2, -3, 6 \rangle \qquad R : \mathbf{-1}x + \mathbf{\frac{3}{2}}y \mathbf{-3}z = 14 \Rightarrow \mathbf{n}_R = \langle -1, \frac{3}{2}, -3 \rangle$$
$$S : \mathbf{6}x + \mathbf{8}y + \mathbf{2}z = 1 \Rightarrow \mathbf{n}_S = \langle 6, 8, 2 \rangle \qquad T : \mathbf{-9}x \mathbf{-12}y \mathbf{-3}z = 7 \Rightarrow \mathbf{n}_T = \langle -9, -12, -3 \rangle$$

Notice that $\mathbf{n}_Q = -2\mathbf{n}_R$. It means that:

$$\frac{2}{-1} = \frac{-3}{3/2} = \frac{6}{-3} = -2$$

This implies that Q and R are parallel. Similarly, $\mathbf{n}_T = -\frac{3}{2}\mathbf{n}_S$,

$$\frac{-9}{6} = \frac{-12}{8} = \frac{-3}{2} = -\frac{3}{2}$$

so S and T are parallel. Furthermore, $\mathbf{n}_Q \cdot \mathbf{n}_S = 0$:

$$\mathbf{n}_Q \cdot \mathbf{n}_S = 2(6) + (-3)(8) + 6(2) = 0$$

and $\mathbf{n}_Q \cdot \mathbf{n}_T = 0$:

$$\mathbf{n}_Q \cdot \mathbf{n}_T = 2(-9) + (-3)(-12) + 6(-3) = 0$$

which implies that Q is orthogonal to S and T. Because Q and R are parallel, it follows that R is also orthogonal to both S and T.