Integral Calculus

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1 Fundamental theorem of Calculus

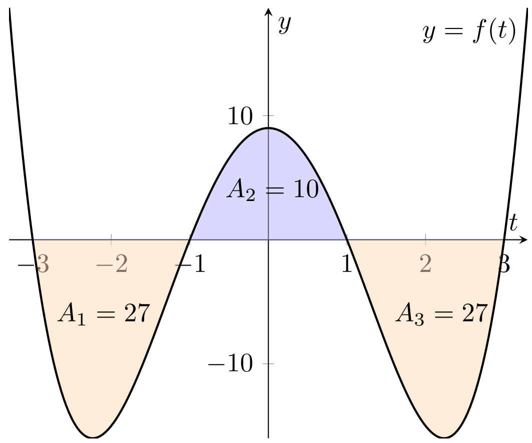
1.1 Area function

Definition 1. Let f be continuous function, for $t \geqslant a$. The **area function for f with left endpoint a** is:

$$A(x) = \int_a^x f(t)dt$$

where $x \geqslant a$. The area function gives the net area of the region bounded by the graph of f and the $t-$axis on the interval $[a, x]$.

Example 1.1. Comparing area functions. The following graph of f has areas at various regions. Let $A(x) = \int_{-3}^{x} f(t)dt$ and $F(x) = \int_{-1}^{x} f(t)dt$ be two area functions for f.



Evaluate the following area functions:

1. $A(-1)$ and $F(-1)$
2. $A(1)$ and $F(1)$
3. $A(3)$ and $F(3)$

Solution:

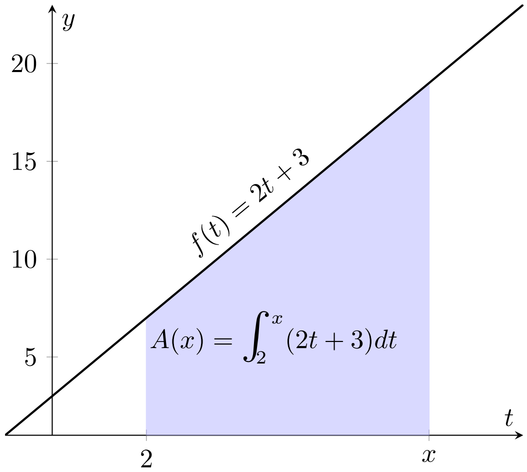
1. The value of $A(-1) = \int_{-3}^{-1} f(t)dt$ is the net area of the region bounded by the graph of f and the $t-$axis on the interval $[-3, -1]$. We see $A_1 = 27$. So $A(-1) = -27$

   On the over hand, $F(-1) = \int_{-1}^{-1} f(t)dt = 0$ (*Property 1*). Notice that $A(-1) - F(-1) = -27$

2. The value of $A(1) = \int_{-3}^{1} f(t)dt$ is found by subtracting the area below the $t-$axis on $[-3, -1]$ and the area above the $t-$axis on $[-1, 1]$. Therefore, we have $A(1) = 10 - 27 = -17$. Similarly, $F(1)$ is the net area of the region bounded by the graph of f and the $t-$axis on $[-1, 1]$. Therefore, $F(1) = 10$. Notice that $A(1) - F(1) = -27$.

3. Reasoning as in part (1) and (2), we see that $A(3) = -27 + 10 - 27 = -44$ and $F(3) = 10 - 27 = -17$. As before observe that $A(3) - F(3) = -27$

Example 1.2. Area of a trapezoid. Consider the trapezoid bounded by $f(t) = 2t + 3$ and the $t-$axis from $t = 2$ to $t = x$. $A(x) = \int_{2}^{x} (2t + 3)dt$ gives the area of the trapezoid, for $x \geqslant 2$



1. Evaluate $A(2)$

2. Evaluate $A(5)$

3. Find and graph the area function $y = A(x)$, for $x \geqslant 2$

4. Compare the derivative of A to f.

Solution:

1. By Property 1 $A(2) = \int_{2}^{2} (2t + 3)dt = 0$

2. $A(5)$ is the area of the trapezoid on the interval $[2, 5]$. Using the area formula of a trapezoid, we have:

$$A(5) = \int_{2}^{5} (2t + 3)dt = \frac{1}{2}(5 - 2) \cdot (f(2) + f(5)) = \frac{1}{2} \cdot 3(7 + 13) = 30$$

3. Now the endpoint is the variable $x \geqslant 2$. The distance between the parallel sides is $x - 2$.

$$A(x) = \frac{1}{2}(x - 2) \cdot (f(2) + f(x)) = \frac{1}{2}(x - 2)(7 + 2x + 3)$$

$$A(x) = \int_{2}^{x} (2t + 3)dt = x^2 + 3x - 10$$

4. Differentiating the area function:

$$A'(x) = \frac{d}{dx}(x^2 + 3x - 10) = 2x + 3 = f(x)$$

Therefore $A'(x) = f(x)$, or equivalently, the area function A is an antiderivative of f. **It is the first part of the Fundamental Theorem of Calculus**

First part of the Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ then the area function $A(x) = \int_a^x f(t)dt$, for $a \leqslant x \leqslant b$ is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies:

$$A'(x) = \frac{d}{dx}A(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Which means that **the area function of f is an antiderivative of f on** $[a, b]$.