Homework week 8 solutions

Daniel Rakotonirina

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1. Evaluate the following integral $\int_{-1}^{2} \frac{5x}{x^2 - x - 6}\, dx$ **(2.5 marks)**

   Solution:

$$\frac{5x}{x^2 - x - 6} = \frac{A_1}{x-3} + \frac{A_2}{x+2} = \frac{A_1(x+2) + A_2(x-3)}{(x-3)(x+2)} \quad \textbf{(1 mark)}$$

   We have:

$$5x = (A_1 + A_2)x + 2A_1 - 3A_2$$

   From that we have a system of 2 equations **(1 mark)**:

$$5 = A_1 + A_2 \tag{1}$$

$$0 = 2A_1 - 3A_2 \tag{2}$$

   The resolution of the system gives $A_1 = 3$ and $A_2 = 2$. Thus,

$$\int_{-1}^{2} \frac{5x}{x^2 - x - 6}\, dx = \int_{-1}^{2} \frac{3}{x-3}\, dx + \int_{-1}^{2} \frac{2}{x+2}\, dx = 3\ln|x-3|\Big|_{-1}^{2} + 2\ln|x+2|\Big|_{-1}^{2}$$

$$\int_{-1}^{2} \frac{5x}{x^2 - x - 6}\, dx = -\ln 4 \quad \textbf{(0.5 mark)}$$

2. Evaluate the following integral $\int_{-\frac{3}{2}}^{\frac{1}{2}} xe^{2x-1}\, dx$ **(2.5 marks)**

   Solution:

   - Integration by part **(1 mark)**:

Functions in original integral	$u = x$	$dv = e^{2x-1}dx$
Functions in new integral	$du = dx$	$v = \frac{1}{2}e^{2x-1}$

     Thus, the integral becomes:

$$\int_{-\frac{3}{2}}^{\frac{1}{2}} xe^{2x-1}\, dx = \frac{x}{2}e^{2x-1}\Big|_{-\frac{3}{2}}^{\frac{1}{2}} - \frac{1}{2}\int_{-\frac{3}{2}}^{\frac{1}{2}} e^{2x-1}\, dx$$

   - Substitution for $\int_{-\frac{3}{2}}^{\frac{1}{2}} e^{2x-1}\, dx$: $w = 2x - 1 \Longrightarrow dw = 2dx$ **(1 mark)**

$$\int_{-\frac{3}{2}}^{\frac{1}{2}} e^{2x-1}\, dx = \int_{-4}^{0} e^w \frac{dw}{2} = \frac{1}{2}e^w\Big|_{-4}^{0} = \frac{1}{2}(1 - e^{-4})$$

Finally, we have **(0.5 mark)**:

$$\int_{-\frac{3}{2}}^{\frac{1}{2}} xe^{2x-1}\ dx = \frac{1}{2} + \frac{3}{4}e^{-4} - \frac{1}{4}(1-e^{-4}) = \frac{1}{4} + e^{-4}$$

3. Evaluate the following integral $\displaystyle\int \frac{\ln(\tan x)}{\sin x\ \cos x}\ dx$ **(2.5 marks)**

Solution:

- Substitution $u = \tan x$. So, $du = \sec^2 x\ dx$. Thus,

$$\int \frac{\ln(\tan x)}{\sin x\ \cos x}\ dx = \int \frac{\ln u}{\sin x\ \cos x}\frac{du}{\sec^2 x} = \int \frac{\ln u}{\sin x\ \cos x}\frac{du}{\frac{1}{\cos^2 x}} = \int \frac{\ln u}{\tan x}\ du = \int \frac{\ln u}{u}\ du \quad \textbf{(1 mark)}$$

- Deal with $\displaystyle\int \frac{\ln u}{u}\ du$ using a second substitution $w = \ln u \Longrightarrow dw = \frac{1}{u}\ du$

$$\int \frac{\ln u}{u}\ du = \int \frac{w}{u}\cdot u\ dw = \int w\ dw = \frac{w^2}{2} + c = \frac{\ln^2 u}{2} + c = \frac{\ln^2(\tan x)}{2} + c \quad \textbf{(1 mark)}$$

Finally,

$$\int \frac{\ln(\tan x)}{\sin x\ \cos x}\ dx = \frac{\ln^2(\tan x)}{2} + c \quad \textbf{(0.5 mark)}$$

4. Evaluate the following integral $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\ dx$ **(2.5 marks)**

Solution:
We have:

$$\int \frac{\sqrt{25x^2-4}}{x}\ dx = 5\int \frac{\sqrt{x^2-\frac{4}{25}}}{x}\ dx$$

We use trigonometric substitution with $x = \frac{2}{5}\sec\theta \Longrightarrow dx = \frac{2}{5}\sec\theta\tan\theta\ d\theta$ **(0.5 mark)**

$$\sqrt{x^2-\frac{4}{25}} = \sqrt{\frac{4}{25}\sec^2\theta - \frac{4}{25}} = \frac{2}{5}\sqrt{\tan^2\theta} = \frac{2}{5}\tan\theta \quad \textbf{(0.5 mark)}$$

So,

$$\begin{aligned}\int \frac{\sqrt{25x^2-4}}{x}\ dx &= 5\int \frac{2/5\tan\theta}{2/5\sec\theta}(2/5\sec\theta\tan\theta\ d\theta)\\ &= 2\int \tan^2\theta\ d\theta = 2\int(\sec^2\theta - 1)\ d\theta\\ &= 2\int \sec^2\theta\ d\theta - 2\int d\theta\\ &= 2\tan\theta - 2\theta + c \quad \textbf{(0.5 mark)}\end{aligned}$$

Since $x = \frac{2}{5}\sec\theta$, we get $\frac{5x}{2} = \sec\theta$ so $\cos\theta = \frac{2}{5x}$. Hence, $\theta = \arccos(\frac{2}{5x})$. **(0.5 mark)**
Finally,

$$\int \frac{\sqrt{25x^2-4}}{x}\ dx = 2\tan\theta - 2\theta + c = 5\sqrt{x^2-\frac{4}{25}} - 2\arccos(\frac{2}{5x}) + c \quad \textbf{(0.5 mark)}$$