Integral Calculus

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1 Trigonometric Integrals (continue)

Sometimes the integrands are not explicitly of the forms $\int \sin^m x \cos^n x \; dx$ and $\int \tan^m x \sec^n x \; dx$ so we have to use some algebraic manipulations.

Example 1.1. Evaluate the following integral $I = \int \frac{2}{\sqrt{x}} \sin^6(\sqrt{x}) \cos^3(\sqrt{x}) \; dx$.

Solution:

- First substitution $u = \sqrt{x}$ and $du = \frac{dx}{2\sqrt{x}}$ to transform this integral into $\int \sin^m x \cos^n x \; dx$. So we have:

$$I = 2\int \frac{1}{\sqrt{x}} \sin^6 u \cos^3 u \; 2\sqrt{x} \; du = 4\int \sin^6 u \cos^3 u \; du = 4\int \sin^6 u \underbrace{\cos^2 u}_{1-\sin^2 u} \cos u \; du$$

- Second substitution $t = \sin u$ and $dt = \cos u \; du$. We have:

$$I = 4\int t^6(1-t^2) \; dt = 4\int (t^6 - t^8) \; dt = 4\left(\frac{t^7}{7} - \frac{t^9}{9}\right) + c$$

- Replace the new variables by their initial values

$$I = 4\left(\frac{t^7}{7} - \frac{t^9}{9}\right) + c = 4\left(\frac{\sin^7 u}{7} - \frac{\sin^9 u}{9}\right) + c = 4\left(\frac{\sin^7(\sqrt{x})}{7} - \frac{\sin^9(\sqrt{x})}{9}\right) + c$$

2 Trigonometric Substitutions

2.1 Integral involving $a^2 - x^2$

We may deal with an integral whose integrand contains the term $a^2 - x^2$, where a is a positive constant. Observe what happens when x is replaced with $a \sin\theta$ so $dx = a\cos\theta \; d\theta$.

$$\begin{aligned} a^2 - x^2 &= a^2 - (a\sin\theta)^2 && \text{Replace } x \text{ with } a\sin\theta \\ &= a^2 - a^2\sin^2\theta && \text{Simplify} \\ &= a^2(1-\sin^2\theta) && \text{Factor} \\ &= a^2\cos^2\theta && 1-\sin^2\theta = \cos^2\theta \end{aligned}$$

Example 2.1. Evaluate the following integral $\int_0^4 \sqrt{16-x^2}\ dx$.

Solution:
Here, $a=4$ so we use substitution $x = a\sin\theta = 4\sin\theta$ then $dx = 4\cos\theta\ d\theta$ and $\theta = \sin^{-1}(\frac{x}{4})$. So, we have:

$$\int_0^4 \sqrt{16-x^2}\ dx = \int_{\sin^{-1}(0)}^{\sin^{-1}(1)} \sqrt{16-(4\sin\theta)^2}\ \underbrace{4\cos\theta\ d\theta}_{dx} = \int_0^{\frac{\pi}{2}} \sqrt{16-16\sin^2\theta}\ 4\cos\theta\ d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{16}\sqrt{(1-\sin^2\theta)}\ \cos\theta\ d\theta = 16\int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$= 16\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\ d\theta = 16\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Big|_0^{\frac{\pi}{2}} = 4\pi$$

Example 2.2. Evaluate the following integral $\int_0^{\frac{3}{2}} \frac{x\ dx}{\sqrt{9-x^2}}$.

Solution:
Here $a=3$ so $x=3\sin\theta$, $dx = 3\cos\theta\ d\theta$ and $\theta = \sin^{-1}(\frac{x}{3})$. We have:

$$\int_0^{\frac{3}{2}} \frac{x\ dx}{\sqrt{9-x^2}} = \int_{\sin^{-1}(0)}^{\sin^{-1}(\frac{1}{2})} \frac{3\sin\theta}{\sqrt{9-9\sin^2\theta}}\ 3\cos\theta\ d\theta = \int_0^{\frac{\pi}{6}} \frac{9\cos\theta\sin\theta\ d\theta}{\sqrt{9(1-\sin^2\theta)}} = \int_0^{\frac{\pi}{6}} \frac{9\cos\theta\sin\theta\ d\theta}{3\cos\theta}$$

$$= 3\int_0^{\frac{\pi}{6}} \sin\theta\ d\theta = -3(\cos\theta)\Big|_0^{\frac{\pi}{6}} = -3(\frac{\sqrt{3}}{2}-1) = 3-\frac{3\sqrt{3}}{2}$$

2.2 Integral involving a^2+x^2

We may deal with an integral whose integrand contains the term a^2+x^2, where a is a positive constant. Observe what happens when x is replaced with $a\tan\theta$ so $dx = a(1+\tan^2\theta)\ d\theta = a\sec^2\theta\ d\theta$.
Note that

$$a^2+x^2 = a^2+a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

Example 2.3. Evaluate the following integral $\int_0^1 \frac{2}{\sqrt{x^2+36}}dx$

Solution:
Substitution: $a=6$, $x=6\tan\theta$, $\theta=\tan^{-1}(\frac{x}{6})$ and $dx = 6\sec^2\theta\ d\theta$. So we have:

$$\int_0^1 \frac{2}{\sqrt{x^2+36}}\ dx = \int_{\tan^{-1}(0)}^{\tan^{-1}(\frac{1}{6})} \frac{2}{\sqrt{(6\tan\theta)^2+36}}\ 6\sec^2\theta\ d\theta = \int_0^{\tan^{-1}(\frac{1}{6})} \frac{12\sec^2\theta}{\sqrt{36(1+\tan^2\theta)}}\ d\theta$$

$$= \int_0^{\tan^{-1}(\frac{1}{6})} \frac{12\sec^2\theta}{6\sec\theta}\ d\theta = 2\int_0^{\tan^{-1}(\frac{1}{6})} \sec\theta\ d\theta = 2\ln|\sec\theta+\tan\theta|\Big|_0^{\tan^{-1}(\frac{1}{6})}$$

$$= 2\ln|\sec(\tan^{-1}(\frac{1}{6}))+\tan(\tan^{-1}(\frac{1}{6}))| - 2\ln|\sec(0)+\tan(0)|$$

$$= 2\ln|\sec(\tan^{-1}(\frac{1}{6}))+\frac{1}{6}|$$

Example 2.4. Evaluate the integral $\int \frac{dx}{x^2\sqrt{9+x^2}}$

Here $a = 3$ so $x = 3\tan\theta$, $dx = 3\sec^2\theta\ d\theta$ and $\theta = \tan^{-1}(\frac{x}{3})$. So we have:

$$\int \frac{dx}{x^2\sqrt{9+x^2}} = \int \frac{3\sec^2\theta\ d\theta}{(3\tan\theta)^2\sqrt{9+(3\tan\theta)^2}} = \int \frac{3\sec^2\theta\ d\theta}{(3\tan\theta)^2\cdot 3\sec\theta} = \frac{1}{9}\int \frac{\sec\theta\ d\theta}{\tan^2\theta}$$

$$= \frac{1}{9}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\ d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\ d\theta$$

Now let $u = \sin\theta \implies du = \cos\theta\ d\theta$. Then,

$$\int \frac{dx}{x^2\sqrt{9+x^2}} = \frac{1}{9}\int \frac{du}{u^2} = -\frac{1}{9u} + c = -\frac{1}{9\sin\theta} + c = -\frac{1}{9\sin(\tan^{-1}(\frac{x}{3}))} + c$$

2.3 Integral involving $x^2 - a^2$

We may deal with an integral whose integrand contains the term $x^2 - a^2$, where a is a positive constant. Observe what happens when x is replaced with $a\sec\theta$ so $dx = a\sec\theta\tan\theta\ d\theta$:

$$x^2 - a^2 = (a\sec\theta)^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$

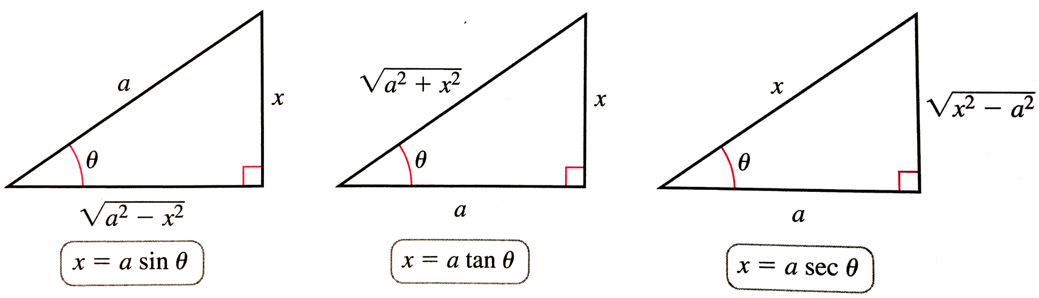
Example 2.5. Evaluate the following integral $\displaystyle\int \frac{dx}{\sqrt{x^2-49}}$.

Here $a = 7$, so $x = 7\sec\theta \implies dx = 7\sec\theta\tan\theta\ d\theta$. So we have:

$$\int \frac{dx}{\sqrt{x^2-49}} = \int \frac{7\sec\theta\tan\theta\ d\theta}{\sqrt{(7\sec\theta)^2-49}} = \int \frac{7\sec\theta\tan\theta\ d\theta}{7\sqrt{\sec^2\theta-1}} = \int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + c$$

$$= \ln|\sec(\sec^{-1}(\frac{x}{7})) + \tan(\sec^{-1}(\frac{x}{7}))| + c$$

Summary



The Integral Contains . . .	Corresponding Substitution	Useful Identity
$a^2 - x^2$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for $\lvert x\rvert \le a$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta, \begin{cases} 0 \le \theta < \frac{\pi}{2}, \text{for } x \ge a \\ \frac{\pi}{2} < \theta \le \pi, \text{for } x \le -a \end{cases}$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

2.4 Algebraic manipulation

Sometimes, the integrand is not explicitly of the forms $a^2 \pm x^2$ and $x^2 - a^2$

Example 2.6. Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{4x^2+25}}$.

Solution:
We make the coefficient of x^2 to be 1. We have $4x^2+25 = 4(x^2+\frac{25}{4}) = 4(x^2+(\frac{5}{2})^2)$. The we use $x = \frac{5}{2}\tan\theta$, $dx = \frac{5}{2}\sec^2\theta\ d\theta$ and $\theta = \tan^{-1}(\frac{2x}{5})$ The we have:

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2+25}} &= \int \frac{dx}{\sqrt{4(x^2+(\frac{5}{2})^2)}} = \int \frac{dx}{2\sqrt{x^2+(\frac{5}{2})^2}} = \frac{1}{2}\int \frac{\frac{5}{2}\sec^2\theta\ d\theta}{\sqrt{(\frac{5}{2}\sec\theta)^2+(\frac{5}{2})^2}} \\ &= \frac{1}{2}\int \frac{\frac{5}{2}\sec^2\theta\ d\theta}{\sqrt{\frac{25}{4}(\tan^2\theta+1)}} = \frac{1}{2}\int \frac{\sec^2\theta\ d\theta}{\sec\theta} = \frac{1}{2}\ln|\sec\theta+\tan\theta| + c \\ &= \frac{1}{2}\ln|\sec(\tan^{-1}(\frac{2x}{5})) + \frac{2x}{5}| + c\end{aligned}$$