Integral Calculus

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Vectors (textbook sec. 11.1 - 11.2 - 11.3)

It gives the position of a point in space (from the origin of the reference frame) or quantities that have both *length* (or *magnitude*) and *direction*. It has a *Tail* and a *Head*.

0.1 2D space and 3D space

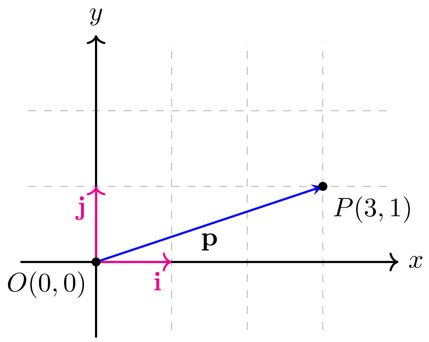


Figure 1: 2D space

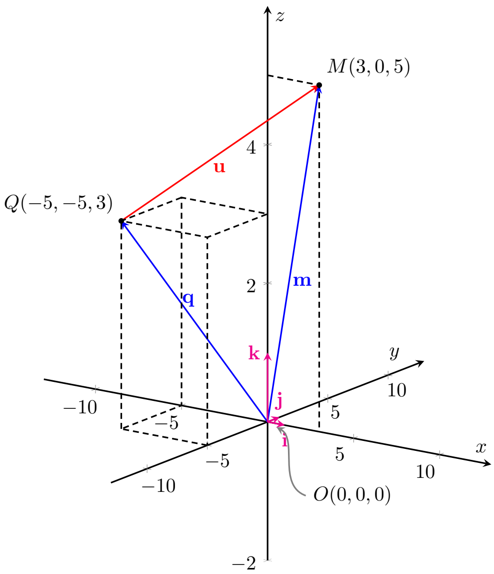


Figure 2: 3D space

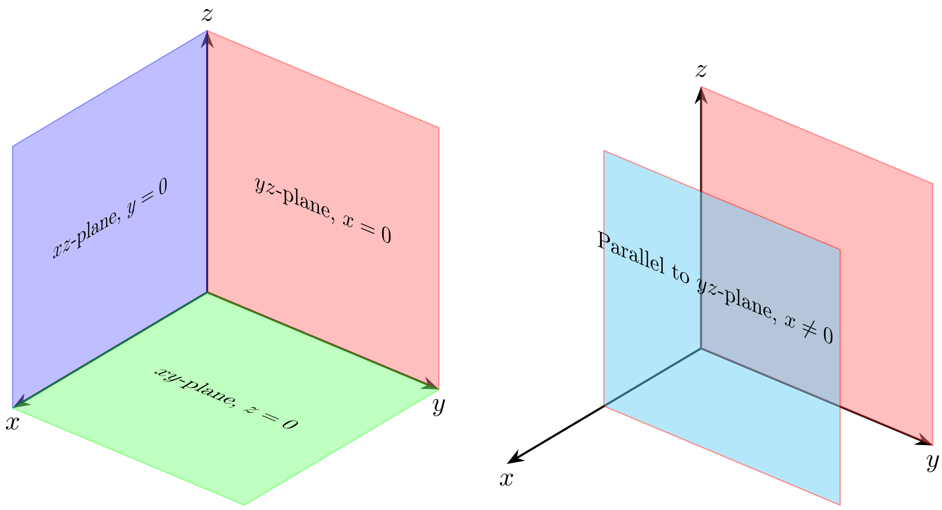
2D space (2 components)	3D space (3 components)
Point	
$(\star,\star)$: $P(P_1,P_2)$	$(\star,\star,\star)$: $M(M_1,M_2,M_3)$
Vector	
$\langle\star,\star\rangle$ or $a\mathbf{i}+b\mathbf{j}$ $\overrightarrow{OP}=\mathbf{p}=\langle P_1-O_1,P_2-O_2\rangle=\langle p_1,p_2\rangle$ $\overrightarrow{OP}=\mathbf{p}=p_1\mathbf{i}+p_2\mathbf{j}$	$\langle\star,\star,\star\rangle$ or $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ $\overrightarrow{OQ}=\mathbf{q}=\langle Q_1-O_1,Q_2-O_2,Q_3-O_3\rangle=\langle q_1,q_2,q_3\rangle$ $\overrightarrow{OQ}=\mathbf{q}=q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}$

Example 0.1. $\overrightarrow{OP}=\mathbf{p}=\langle 3-0,1-0\rangle=\langle 3,1\rangle=3\mathbf{i}+1\mathbf{j}=3\mathbf{i}+\mathbf{j}$

Example 0.2. $\overrightarrow{QM}=\mathbf{u}=\langle 3-(-5),0-(-5),5-3\rangle=\langle 8,5,2\rangle=8\mathbf{i}+5\mathbf{j}+2\mathbf{k}$

0.2 Planes

A plane is a flat surface, two-dimensional surface that extends infinitely far. The world has three dimensions, but a plane has two dimensions and spanned by two non parallel vectors.



Related Exercises sec. 11.2 (15–22)

0.3 Operations

0.3.1 Length or magnitude

The length or the magnitude of a given vector $\mathbf{a} = \overrightarrow{AB} = \langle a_1, a_2, a_3 \rangle$ is:

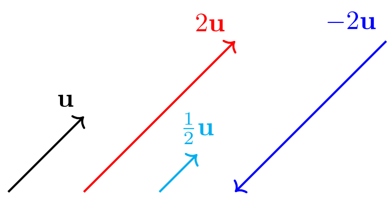
$$|\mathbf{a}| = |\overrightarrow{AB}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Related Exercises sec. 11.1 (23–27)

0.3.2 Multiplication by a scalar (number)

Let c a scalar (number) and $\mathbf{v}$ a vector. The resulting vector of the multiplication of $\mathbf{v}$ by c is denoted $c\mathbf{v}$. It is called a *scalar multiple* of $\mathbf{v}$. If $c > 0$, $c\mathbf{v}$ has the same direction as $\mathbf{v}$, otherwise it has the opposite direction. The operation is written as follows:

$$c \cdot \mathbf{v} = \langle c \cdot v_1, c \cdot v_2, c \cdot v_3 \rangle$$



Example 0.3. $2 \cdot \langle 2, 1, -3 \rangle = \langle 4, 2, -6 \rangle$

Note: Two vectors $\mathbf{u} = \langle \mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3} \rangle$ and $\mathbf{v} = \langle \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \rangle$ are parallel if:

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3} = c$$

Example 0.4. Let $\mathbf{u} = \langle 2, 1, -3 \rangle$ and $\mathbf{v} = \langle 4, 2, -6 \rangle$ be two vectors. Determine if their are parallel or not. $\frac{2}{4} = \frac{1}{2} = \frac{-3}{-6} = \frac{1}{2}$. So $\mathbf{u}$ and $\mathbf{v}$ are parallel.

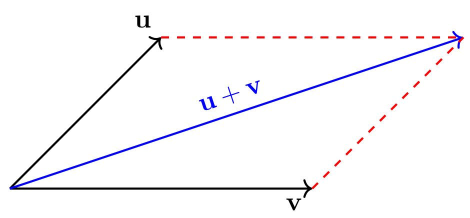
Related Exercises sec. 11.1 (17–20)

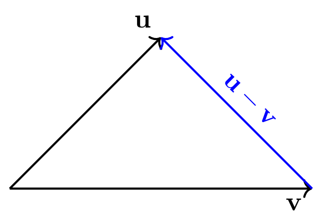
0.3.3 Addition and subtraction

Let $\mathbf{u}$ and $\mathbf{v}$ two vectors.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$





Related Exercises sec. 11.1 (21–22)

0.3.4 Dot product

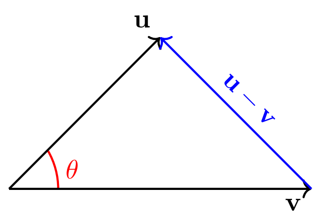
Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle u_1, u_2, u_3 \rangle$ be two vectors. The dot product is defined as:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{1}$$

OR

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cos\theta \tag{2}$$

It is a number (or scalar), NOT a vector! *Related Exercises sec. 11.3 (15–24)*



Eq. (2) gives implicitly the angle θ between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leqslant \theta \leqslant \pi$. θ is obtained by:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} \Longrightarrow \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|}\right)$$

Note: Two vectors $\mathbf{u} = \langle \mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3} \rangle$ and $\mathbf{v} = \langle \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \rangle$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$. From Eq. (2) $\cos\theta = 0$, which gives $\theta = \frac{\pi}{2}$. *Related Exercises sec. 11.3 (9–14)*

Example 0.5. Find the dot product of $\mathbf{u} = \langle 1, 3, -2\rangle$ and $\mathbf{v} = \langle 4, -1, 0\rangle$.
$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 4 + 3 \cdot (-1) + (-2) \cdot 0 = 1$

Example 0.6. Compute the angle between $\mathbf{u} = \langle \sqrt{3}, 1, 0\rangle$ and $\mathbf{v} = \langle 1, \sqrt{3}, 0\rangle$.

- $\mathbf{u} \cdot \mathbf{v} = \sqrt{3} \cdot 1 + 1 \cdot \sqrt{3} + 0 \cdot 0 = 2\sqrt{3}$
- $|\mathbf{u}| = \sqrt{(\sqrt{3})^2 + 1^2 + 0^2} = \sqrt{3+1} = \sqrt{4} = 2$
- $|\mathbf{v}| = \sqrt{1^2 + (\sqrt{3})^2 + 0^2} = \sqrt{1+3} = \sqrt{4} = 2$
- $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \dfrac{2\sqrt{3}}{2 \cdot 2} = \dfrac{\sqrt{3}}{2} \Longrightarrow \theta = \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{6}$

Properties of the dot product

Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are 3 vectors and let c be a scalar (or a number).

Theorem 1. Commutativity property. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

Example 0.7. Let $\mathbf{u} = \langle 1, 3, -2\rangle$ and $\mathbf{v} = \langle 4, -1, 0\rangle$ be two vectors.
$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 4 + 3 \cdot (-1) + (-2) \cdot 0 = 1$
$\mathbf{v} \cdot \mathbf{u} = 4 \cdot 1 + (-1) \cdot 3 + 0 \cdot (-2) = 1$

Theorem 2. Associativity property. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

Example 0.8. Let $\mathbf{u} = \langle 1, 3, -2\rangle$ and $\mathbf{v} = \langle 4, -1, 0\rangle$ be two vectors.
$2(\mathbf{u} \cdot \mathbf{v}) = 2(1 \cdot 4 + 3 \cdot (-1) + (-2) \cdot 0) = 2$
$2\mathbf{u} \cdot \mathbf{v} = \langle 2, 6, -4\rangle \cdot \langle 4, -1, 0\rangle = 2 \cdot 4 + 6 \cdot (-1) + (-4) \cdot 0 = 2$
$\mathbf{u} \cdot 2\mathbf{v} = \langle 1, 3, -2\rangle \cdot \langle 8, -2, 0\rangle = 1 \cdot 8 + 3 \cdot (-2) + (-2) \cdot 0 = 2$

Theorem 3. Distributive property. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$