Homework week 11 solutions

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1. A continuous random variable X is given by the following probability density function

$$f(x) = \begin{cases} \frac{1}{4} + \frac{1}{2}|x| & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the expected value $E(X)$ of the random variable X (**1 mark**)
   Solution:

$$E(X) = \int_{-\infty}^{+\infty} x\ f(x)dx = \int_{-\infty}^{+\infty} x \left(\frac{1}{4} + \frac{1}{2}|x|\right)\ dx = \int_{-1}^{1} x \left(\frac{1}{4} + \frac{1}{2}|x|\right)\ dx$$
$$= \frac{1}{4}\int_{-1}^{1} x\ dx + \frac{1}{2}\int_{-1}^{1} x|x|\ dx = 0$$

   (b) Let $F(x)$ be the cumulative distribution function for the random variable X. Find $F(x)$ for $0 < x < 1$. (**1 mark**)
   Solution:

   - for $x \in]-\infty; -1[$

$$F(x) = \int_{-\infty}^{-1} f(x)dx = 0$$

   - for $x \in]-1; 1[$

$$F(x) = \int_{-1}^{1} f(x)dx = \frac{1}{4}\int_{-1}^{1} dx + \frac{1}{2}\int_{-1}^{1} |x|\ dx = 1$$

   - for $x \in]1; +\infty[$

$$F(x) = \int_{1}^{+\infty} f(x)dx = 0$$

2. Is there any value of k for which the function f below is a probability density function? (**2 marks**)

$$f(x) = \begin{cases} \frac{2k}{(k+x)(k-x)} & \text{for } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

   If yes, find all such values of k. If there is no such k, explain why.
   Solution:
   First we integrate the function over $]-\infty; +\infty[$ (definition of a PDF) so we have:

$$\int_{-\infty}^{+\infty} \frac{2k}{(k+x)(k-x)}\ dx = \int_{0}^{\frac{1}{2}} \frac{2k}{(k+x)(k-x)}\ dx = 1$$

Here we use partial fraction:

$$\frac{2k}{(k+x)(k-x)} = \frac{A}{k+x} + \frac{B}{k-x}$$

From that we find that $A = B = 1$. So the integral becomes:

$$\int_0^{\frac{1}{2}} \frac{2k}{(k+x)(k-x)}\, dx = \int_0^{\frac{1}{2}} \frac{1}{k+x}\, dx + \int_0^{\frac{1}{2}} \frac{1}{k-x}\, dx = \ln|k+x|\Big|_0^{\frac{1}{2}} - \ln|k-x|\Big|_0^{\frac{1}{2}} = 1$$

$$= \ln\left|\frac{k+x}{k-x}\right|\Big|_0^{\frac{1}{2}} = \ln\left|\frac{k+\frac{1}{2}}{k-\frac{1}{2}}\right| = 1 \Longrightarrow \frac{k+\frac{1}{2}}{k-\frac{1}{2}} = e \Longrightarrow k = \frac{1}{2}\left(\frac{e+1}{e-1}\right) \simeq 1.08$$

3. Do the following series converge or diverge (**solutions on next page**)?

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}\sqrt{n+1}}$ (**2 marks**)

   (b) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{-1}{5}\right)^n$ (**2 marks**)

   (c) $\displaystyle\sum_{n=10}^{\infty} \cos(\pi n)$ (**2 marks**)

Solutions for Quiz 10

① We can observe that for large k,

$$\frac{1}{\sqrt{k}\sqrt{k+1}} \approx \frac{1}{\sqrt{k}\sqrt{k}} = \frac{1}{k}$$

This is how we write the solution:

So we use $\sum_{k=1}^{\infty} \frac{1}{k}$ as a comparison test and write

$$a_k = \frac{1}{k} \qquad \left(\sum a_k = \sum \frac{1}{k} \right)$$

$$b_k = \frac{1}{\sqrt{k}\sqrt{k+1}} \qquad \left(\sum b_k = \sum \frac{1}{\sqrt{k}\sqrt{k+1}} \right)$$

Limit comparison test:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1/k}{1/(\sqrt{k}\sqrt{k+1})} = \lim_{k\to\infty} \frac{\sqrt{k}\sqrt{k+1}}{k} = \lim_{k\to\infty} \frac{\sqrt{k}\sqrt{k}}{k} = 1 > 0$$

Since $\sum a_k = \sum \frac{1}{k}$ diverges, so does $\sum b_k = \sum \frac{1}{\sqrt{k}\sqrt{k+1}}$.

② This is a geometric series with $r = \frac{-1}{5}$ whose first term is $\left(\frac{-1}{5}\right)^3$:

$$\sum_{n=3}^{\infty} \left(\frac{-1}{5}\right)^n = \left(\frac{-1}{5}\right)^3 + \left(\frac{-1}{5}\right)^4 + \left(\frac{-1}{5}\right)^5 + \dots$$

Since $|r| = \left|\frac{-1}{5}\right| = \frac{1}{5} < 1$, it converges and

$$\sum_{n=3}^{\infty} \left(\frac{-1}{5}\right)^n = \frac{\left(\frac{-1}{5}\right)^3}{1-\left(\frac{-1}{5}\right)} = \frac{\left(\frac{-1}{5}\right)^3}{\frac{6}{5}} = \frac{-1}{5^3}\cdot\frac{5}{6} = \frac{-1}{150}.$$

③ We have $\lim_{n\to\infty} \cos(n\pi) = \lim_{n\to\infty} (-1)^n$ does not exist

Remark: $\cos n\pi = (-1)^n = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$

So, using the divergence test, we see that $\sum_{n=7}^{\infty} \cos(n\pi)$ diverges.

④ We use Ratio test. Let $a_k = \frac{e^k}{k!}$, then:

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{e^{k+1}/(k+1)!}{e^k/k!} = \lim_{k\to\infty} \frac{e\,(k!)}{(k+1)!} = \lim_{k\to\infty} \frac{e}{k+1} = 0 < 1$$

Note 1: $a_k = \frac{e^k}{k!} > 0$, so we could use the Ratio test.

Note 2: $(k+1)! = (k+1)\,k!$, so $\frac{(k+1)!}{k!} = k+1$.

Since $r = 0 < 1$, the Ratio test says that $\sum_{k=1}^{\infty} \frac{e^k}{k!}$ converges.