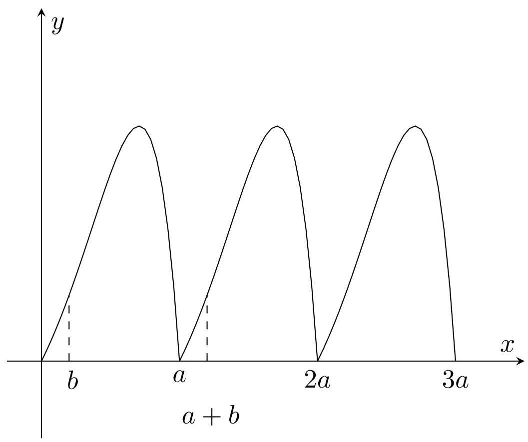
MATH 105-951 – Written Assignment #1

Solution to the bonus question

Question: A function $f(x)$ is *periodic* with *period* a if $f(x+a) = f(x)$ for all x. For example, $f(x) = \sin(x)$ is periodic with period 2π. If $f(x)$ is periodic with period a and continuous on $[0, a]$, show that

$$\int_0^a f(x)dx = \int_b^{a+b} f(x)dx.$$

Solution: The key to solving this problem is drawing a picture. Suppose the graph below represents our periodic function:



From the picture we can see that the area between $x = b$ and $x = a + b$ can be broken into two pieces:

$$\int_b^{a+b} f(x)dx = \int_b^a f(x)dx + \int_a^{a+b} f(x)dx$$

(note that it is NOT necessary for b to be between a and $2a$ for this to be true). From the picture we can also see that the area between $x = a$ and $x = a + b$ should be the same as the area from $x = 0$ to $x = b$. To prove this, let $u = x - a$. Then $du = dx$, $u = 0$ when $x = a$, and $u = b$ when $x = a + b$. So,

$$\int_a^{a+b} f(x)dx = \int_0^b f(u+a)du.$$

Since $f(x)$ is periodic with period a, $f(u + a) = f(u)$ for all u. Therefore

$$\int_a^{a+b} f(x)dx = \int_0^b f(x)dx$$

(note that we are able to change the u to x in the above because u is simply a dummy variable – the integral does not depend on u). So

$$\int_b^{a+b} f(x)dx = \int_b^a f(x)dx + \int_a^{a+b} f(x)dx = \int_b^a f(x)dx + \int_0^b f(x)dx = \int_0^a f(x)dx.$$