Written Assignment #3 — Solutions

(a)
1. If T is the amount of time we spend in the self-serve kiosk, then the amount of time we'd expect to wait in line is $\mathbb{E}(T)$.

$$\mathbb{E}(T) \overset{\text{by def'n}}{=} \int_{-\infty}^{\infty} t\cdot f(t)\,dt = \int_{-\infty}^{0} t\cdot \cancelto{0}{f(t)}\,dt + \int_{0}^{\infty} t\cdot f(t)\,dt$$

$$= \int_0^{\infty} t\cdot \frac{1}{7} e^{-t/7}\,dt$$

IBP: Let $u = t$, $dv = \frac{1}{7}e^{-t/7}dt$; $du = dt$, $v = -e^{-t/7}$

$$= \left(-te^{-t/7}\Big|_0^{\infty}\right) - \int_0^{\infty} -e^{-t/7}\,dt$$

$$= \underbrace{\left(\lim_{t\to\infty} -te^{-t/7}\right)}_{-\infty\cdot 0 \to \text{ indeterminate form}} - 0 - \left(7e^{-t/7}\Big|_0^{\infty}\right)$$

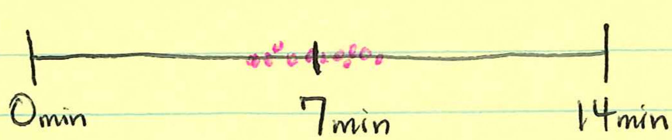
$$= \underbrace{\left(\lim_{t\to\infty} \frac{-t}{e^{t/7}}\right)}_{\frac{-\infty}{\infty} \to \text{ appropriate form for L'Hôpital's Rule}} - \left[\left(\lim_{t\to\infty} \cancelto{0}{7e^{-t/7}}\right) - 7e^{0}\right]$$

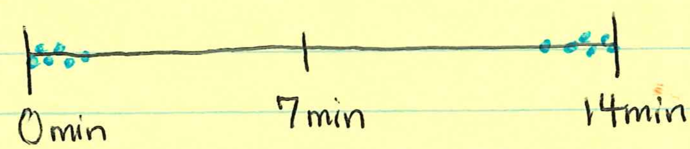
$$= \overbrace{\left(\lim_{t \to \infty} \frac{-1}{\frac{1}{7} e^{t/7}} \right)}^{\text{after applying L'Hôpital's Rule.}} + 7 = 0 + 7 = 7 \text{ (min)}.$$

(b) Let $\tilde{T}$ represent the amount of time we wait in line in the cashier's line.

We are told that the amount we'd expect to wait in either line is the same. This means that $\mathbb{E}(\tilde{T}) = \mathbb{E}(T) = 7$ (by (a))

We are asked to determine which line's wait time is more consistent, so we need to figure out if we'd expect the data to be well-represented by the mean (7 min), or not:





Average of data is 7min and data is well-represented by the expected value.

Average of data is 7min, but data is NOT well-represented by the expected value.

To determine this, we need to calculate the standard deviations of T and $\tilde{T}$. Now,

$$\mathrm{Var}(T) = \mathbb{E}(T^2) - \left[\mathbb{E}(T)\right]^2$$

and

$$\mathbb{E}(T^2) = \int_{-\infty}^{\infty} t^2 \cdot f(t)\,dt = \int_{-\infty}^{0} t^2 \cdot \cancelto{0}{f(t)}\,dt + \int_{0}^{\infty} t^2 \cdot f(t)\,dt$$

$$= \int_0^\infty t^2 \cdot \frac{1}{7} e^{-t/7}\,dt$$

IBP: Let

$u = t^2$ $\quad dv = \frac{1}{7} e^{-t/7}\,dt$

$du = 2t\,dt$ $\quad v = -e^{-t/7}$

$$= \left(-t^2 e^{-t/7} \Big|_0^\infty \right) - \int_0^\infty -2t e^{-t/7}\,dt$$

$$= \left(\lim_{t\to\infty} -t^2 e^{-t/7} \right) - 0 + \int_0^\infty 2t e^{-t/7}\,dt$$

$\underbrace{\quad}_{-\infty \cdot 0}$ indeterminate form

IBP again: Let

$u = 2t$ $\quad dv = e^{-t/7}\,dt$

$du = 2\,dt$ $\quad v = -7e^{-t/7}$

$$= \left(\lim_{t\to\infty} \frac{-t^2}{\underbrace{e^{t/7}}_{\frac{-\infty}{\infty}}} \right) + \left[\left(-14te^{-t/7} \Big|_0^\infty \right) - \int_0^\infty -14e^{-t/7}\,dt \right]$$

$$\overset{L'H}{=} \left(\lim_{t\to\infty} \frac{-2t}{\frac{1}{7}e^{t/7}} \right) + \left(\lim_{t\to\infty} -14te^{-t/7} \right) - 0 - \left(98e^{-t/7} \Big|_0^\infty \right)$$

$$\overset{L'H}{=} \left(\lim_{t\to\infty} \cancelto{0}{\frac{-2}{\frac{1}{49}e^{t/7}}} \right) + \left(\lim_{t\to\infty} \cancelto{0}{\frac{-14t}{e^{t/7}}} \right) - \left[\left(\cancelto{0}{\lim_{t\to\infty} 98e^{-t/7}} \right) - 98e^{0} \right]$$

as in (a)

$$= 98.$$

$$\therefore \operatorname{Var}(T) = \mathbb{E}(T^2) - [\mathbb{E}(T)]^2$$

$$= 98 - 49$$

$$= 49$$

Therefore, the standard deviation of T is

$$\sigma(T) = \sqrt{\operatorname{Var}(T)} = \sqrt{49} = 7 \text{ min.}$$

Now, we are given that $\int_{-\infty}^{\infty} t^2 \cdot g(t)\,dt = 50$. Therefore

$$\mathbb{E}(\tilde{T}^2) = \int_{-\infty}^{\infty} t^2 \cdot g(t)\,dt = 50$$

$$\Rightarrow \operatorname{Var}(\tilde{T}) = \mathbb{E}(\tilde{T}^2) - [\mathbb{E}(T)]^2$$

$$= 50 - 49 = 1$$

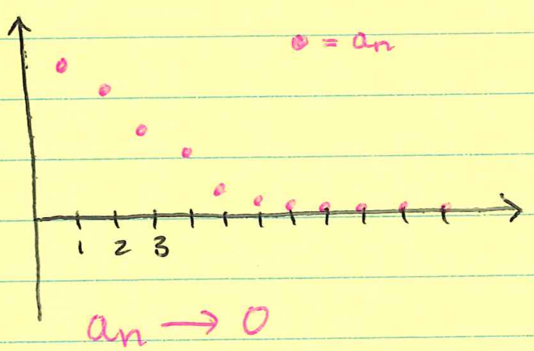
Therefore, the standard deviation of $\tilde{T}$ is

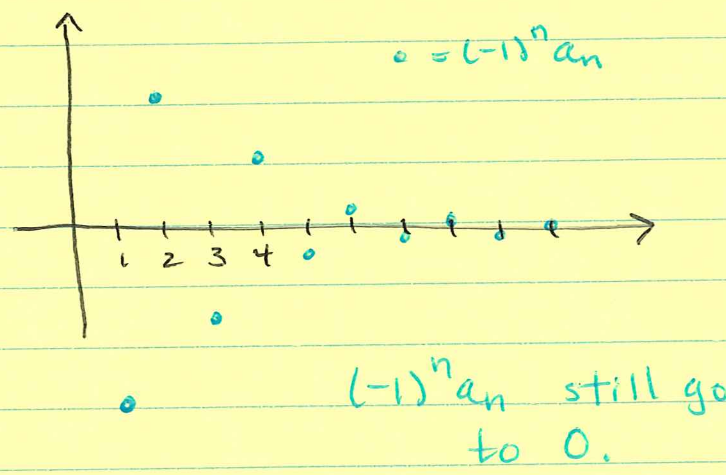
$$\sigma(\tilde{T}) = \sqrt{Var(\tilde{T})} = \sqrt{1} = 1 \text{ min}$$

Since $\sigma(\tilde{T}) = 1$ min, we'd expect that, on average, our wait time in the cashier's line will be within 1 min of 7 minutes (i.e. 6-8 min). For the self-serve kiosk, we'd expect it to be within 7 min of 7 minutes (i.e. 0-14 min).

Therefore, the cashier's line is more consistent.

② (a) This statement is TRUE.





Formally, since $a_n \geq 0$, we have that

$$-a_n \leq (-1)^n a_n \leq a_n \quad \text{for all } n.$$

$$\therefore \underbrace{\lim_{n\to\infty} -a_n}_{= -0 = 0} \leq \lim_{n\to\infty} (-1)^n a_n \leq \underbrace{\lim_{n\to\infty} a_n}_{= 0}$$

So $0 \leq \lim\limits_{n\to\infty} (-1)^n a_n \leq 0 \Rightarrow \lim\limits_{n\to\infty} (-1)^n a_n = 0$

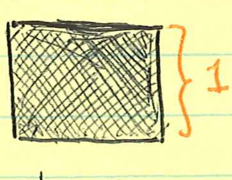
(b) This statement is <u>false</u>.

For example, the sequence $\{(-1)^n\} = \{-1, 1, -1, 1, -1, 1, \ldots\}$ diverges since $\lim\limits_{n\to\infty} (-1)^n$ DNE, and $\lim\limits_{n\to\infty} (-1)^n \neq \pm\infty$.

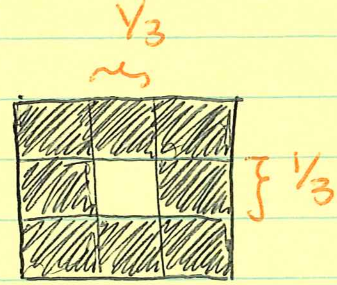
(c) This statement is <u>false</u>.

Since $\sum\limits_{n=1}^{\infty} a_n$ converges, by Divergence Test we must have that $\lim\limits_{n\to\infty} a_n = 0$. However, this means that $\lim\limits_{n\to\infty} \cos(a_n) = \cos(0) = 1 \neq 0$ (since $\cos(x)$ is a continuous function). Thus $\sum\limits_{n=1}^{\infty} \cos(a_n)$ must diverge by the Divergence Test.

3 (a) We have that

$I_0 =$ 

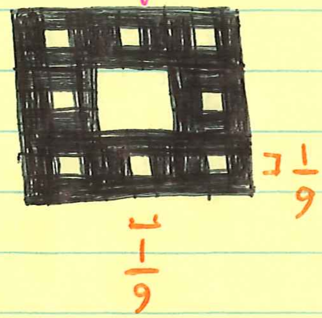
area removed = 0

$I_1 =$ 

area removed $= \underbrace{0}_{\text{from 0th step}} + \underbrace{1}_{\text{\# squares removed}} \cdot \left(\frac{1}{9}\right)$ area of each square removed.

8 little squares

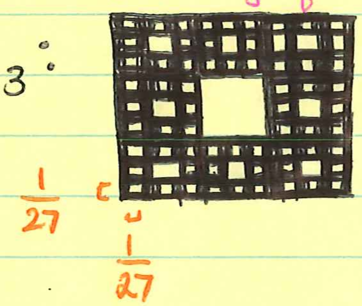
I_2:



$$\text{area removed} = \underbrace{0 + 1\cdot\left(\frac{1}{9}\right)}_{\text{area removed in previous steps}} + \overset{\text{\# of squares removed}}{8}\cdot\underbrace{\left(\frac{1}{9}\right)^2}_{\text{area of each square removed.}}$$

8·8 tiny squares

I_3:



$$\text{area removed} = \underbrace{0 + 1\left(\frac{1}{9}\right) + 8\left(\frac{1}{9}\right)^2}_{\text{area removed in previous steps}} + \underbrace{8^2}_{\text{\# squares removed}}\cdot\underbrace{\left(\frac{1}{9}\right)^3}_{\text{area of each square removed}}$$

From this we can see that in the n^{th} step we will remove

$$8^{n-1}\cdot\left(\frac{1}{9}\right)^n$$

$\therefore$ the total area removed is

$$\sum_{n=1}^{\infty} 8^{n-1}\left(\frac{1}{9}\right)^n = \sum_{n=1}^{\infty} \frac{1}{9}\cdot\left(\frac{8}{9}\right)^{n-1}$$

Geometric series with $a = \frac{1}{9}$ $r = \frac{8}{9}$.

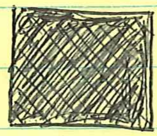
$$= \frac{1/9}{1 - 8/9} = 1$$

(b) We'd expect the total area removed to be finite because we are starting with a square of area 1 and removing bits of area from it — you can't remove more area than what you started with, so the total area removed cannot exceed 1.

(c) [BONUS]

We have that:

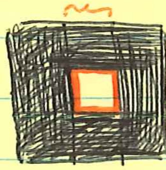
I_0



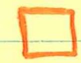
perimeter of square removed $= 0$

I_1:

1/3

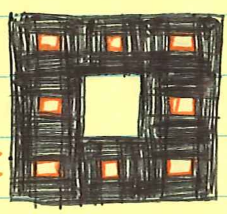


1/3



perimeter of square removed $= \frac{4}{3} = \underbrace{4}_{\#\text{ sides}} \cdot \underbrace{\frac{1}{3}}_{\text{length of each side}}$

I_2:



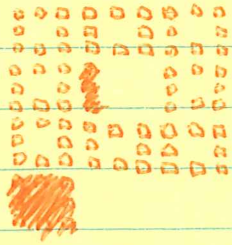
$\frac{1}{9}$

$\frac{1}{9}$

perimeter of squares removed $= \underbrace{8}_{\#\text{ squares}} \cdot \underline{4} \cdot \underbrace{\frac{1}{9}}_{\text{length of each side}} = \underline{8} \cdot \underline{4} \cdot \underbrace{\frac{1}{3^2}}$

I_3: (Pretend I drew it again :))

$\frac{1}{27}$



oops! →

Perimeter of squares removed $= 8^2 \cdot 4 \cdot \frac{1}{27} = 8^2 \cdot 4 \cdot \frac{1}{3^3}$

So, in the n^{th} step, the perimeter of the squares removed is

$$P_n = 4 \cdot 8^{n-1} \cdot \frac{1}{3^n} = \frac{4}{3} \cdot \left(\frac{8}{3}\right)^{n-1}.$$

Therefore, $\lim_{n \to \infty} P_n = \lim_{n \to \infty} \frac{4}{3} \cdot \left(\frac{8}{3}\right)^{n-1} = \infty.$