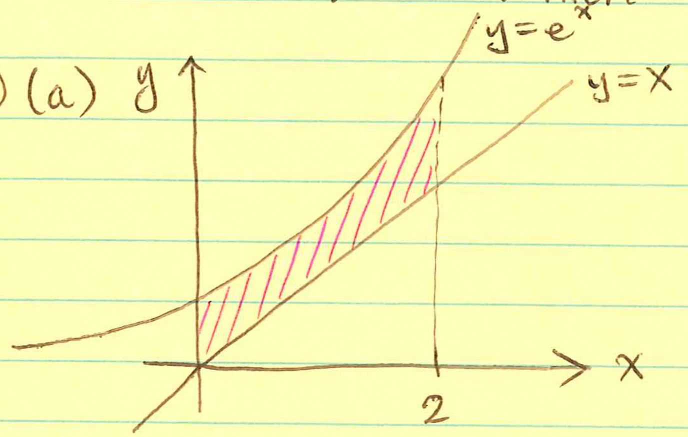
MATH 105-951 Written Assignment #1 - Solutions

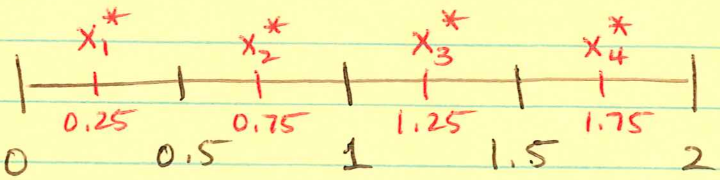
1 (a)



Here $a=0, b=2, n=4$, so:

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$x_k^* = 0 + \left(k-\tfrac{1}{2}\right)\cdot\tfrac{1}{2} = \frac{2k-1}{4}$$



Area between $y=e^x$ and $y=x$ from $x=0$ to $x=2$

$$= \int_0^2 \underbrace{(e^x - x)}_{f(x)}\,dx \approx \sum_{k=1}^{4}\left(e^{\frac{2k-1}{4}} - \left(\frac{2k-1}{4}\right)\right)\cdot\left(\frac{1}{2}\right)$$

$$= \left(e^{0.25} - 0.25\right)\left(\tfrac{1}{2}\right) + \left(e^{0.75} - 0.75\right)\cdot\left(\tfrac{1}{2}\right) + \left(e^{1.25} - 1.25\right)\left(\tfrac{1}{2}\right) + \left(e^{1.75} - 1.75\right)\left(\tfrac{1}{2}\right)$$

$$= 4.322\ldots$$

(b) $|\text{error}| \le \dfrac{M\cdot(b-a)^3}{24n^2}$ $\left(\begin{array}{l} a=0,\ b=2,\ n=4. \\ |f''(x)| \le M. \end{array}\right)$

Now,

$f(x) = e^x - x$, so $f'(x) = e^x - 1 \Rightarrow f''(x) = e^x > 0$

$\Rightarrow |f''(x)| = e^x$

Since e^x is an increasing function, it attains its maximum value at the right endpoint of the interval (i.e. at $x=2$).

$\therefore |f''(x)| = e^x \le e^2$

$$\Rightarrow |\text{error}| \le \frac{e^2(2-0)^3}{24\cdot 4^2} = \frac{e^2\cdot 8}{24\cdot 16} = \frac{e^2}{48} \approx$$

(c) We want $|\text{error}| < 0.001$. From the error bound formula, we know that

$$|\text{error}| \le \frac{e^2 \cdot (2-0)^3}{24 n^2} \quad \Big\} \text{ same values for } a, b \text{ and } M \text{ as in part (b).}$$

So, we ~~want~~ it suffices that

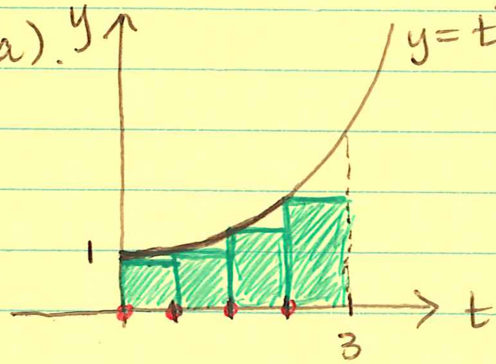
$$\frac{e^2 \cdot 8}{24 n^2} < 0.001$$

$$\Leftrightarrow \quad \frac{e^2 \cdot 8}{24 \cdot (0.001)} < n^2.$$

$$\Rightarrow \quad n > \sqrt{\frac{e^2 \cdot 8}{(24)(0.001)}} \approx 49.63\ldots$$

Therefore we should use at least 50 rectangles.
(n should be an integer, so round up).

(2)(a).

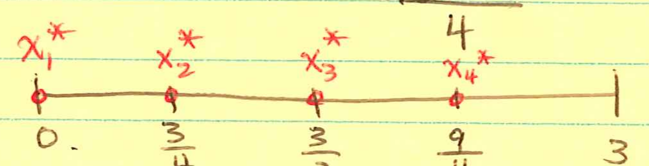


Here: $a=0$, $b=3$ & $n=4$ so:

$$\Delta x = \frac{3-0}{4} = \frac{3}{4}$$

$$x_k^* = x_{k-1} = 0 + (k-1)\Delta x \quad \Big] \text{ left endpoints}$$

$$= \frac{3(k-1)}{4}$$



Therefore,

$$\begin{array}{l}\text{Area under} \\ y=t^2+1 \text{ from} \\ t=0 \text{ to } t=3\end{array} = \int_0^3 \underbrace{(t^2+1)}_{f(t)}\,dt$$

$$\approx \sum_{k=1}^{4} \left(\left(\frac{3(k-1)}{4}\right)^2 + 1 \right)\left(\frac{3}{4}\right) = (0^2+1)\cdot\left(\frac{3}{4}\right) + \left(\left(\frac{3}{4}\right)^2+1\right)\left(\frac{3}{4}\right)$$

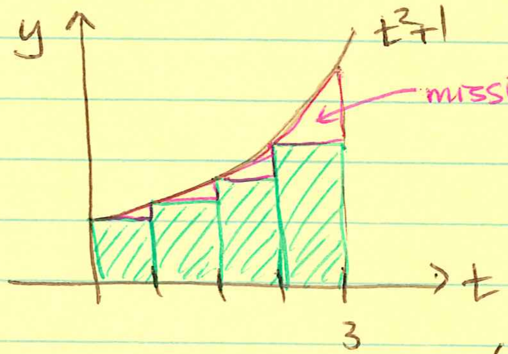
$$+ \left(\left(\frac{3}{2}\right)^2+1\right)\left(\frac{3}{4}\right) + \left(\left(\frac{9}{4}\right)^2+1\right)\left(\frac{3}{4}\right)$$

Rewritten from bottom page 2

$$= (0^2+1)\left(\frac{3}{4}\right) + \left(\left(\frac{3}{4}\right)^2+1\right)\left(\frac{3}{4}\right) + \left(\left(\frac{3}{2}\right)^2+1\right)\left(\frac{3}{4}\right) + \left(\left(\frac{9}{4}\right)^2+1\right)\left(\frac{3}{4}\right)$$

$$\approx \frac{285}{32} \approx 8.90625$$

Because $f(t) = t^2+1$ is an increasing function, when we use the Left Riemann Sum, our rectangles don't cover all of the area under the graph. Therefore, this is an underestimate.



(b) Now, for right endpoints ($\Delta x = 3/4$ as before).

$$x_k^* = x_k = 0 + k\Delta x = \frac{3k}{4}$$

Therefore,

Area under $y = t^2+1$ from $t=0$ to $t=3$

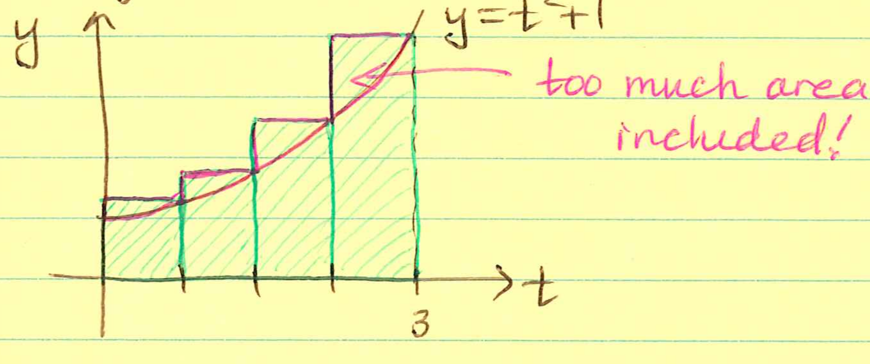
$$= \int_0^3 (t^2+1)\,dt$$

$$\approx \sum_{k=1}^{4} \left(\left(\frac{3k}{4}\right)^2+1\right)\cdot\left(\frac{3}{4}\right)$$

$$= \left(\left(\frac{3}{4}\right)^2+1\right)\left(\frac{3}{4}\right) + \left(\left(\frac{3}{2}\right)^2+1\right)\left(\frac{3}{4}\right) + \left(\left(\frac{9}{4}\right)^2+1\right)\left(\frac{3}{4}\right) + \left((3)^2+1\right)\left(\frac{3}{4}\right)$$

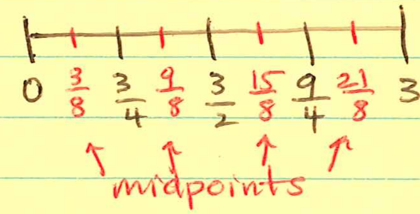
$$= \frac{501}{32} \approx 15.65625$$

Because $f(t) = t^2+1$ is an increasing function, using the Right Riemann sum results in an overestimate as all of the rectangles cover more than just the area under the graph.



(c) For the midpoint Riemann Sum $\left(\Delta x = \frac{3}{4} \text{ as before}\right)$

$$x_k^* = 0 + \left(k - \tfrac{1}{2}\right)\Delta x = \frac{3(2k-1)}{8}$$



So,

Area under $y=t^2+1$ from $t=0$ to $t=3$ $= \displaystyle\int_0^3 (t^2+1)\,dt$

$$\approx \sum_{k=1}^{4}\left(\left(\frac{3(2k-1)}{8}\right)^2+1\right)\left(\frac{3}{4}\right).$$

$$= \left(\left(\frac{3}{8}\right)^2+1\right)\left(\frac{3}{4}\right) + \left(\left(\frac{9}{8}\right)^2+1\right)\left(\frac{3}{4}\right) + \left(\left(\frac{15}{8}\right)^2+1\right)\left(\frac{3}{4}\right)$$
$$+ \left(\left(\frac{21}{8}\right)^2+1\right)\left(\frac{3}{4}\right)$$

$$= \frac{759}{64} = 11.859375$$

We have that $|\text{error}| \le \dfrac{M(b-a)^3}{24n^2}$ $\left(\begin{array}{c} b=3,\ a=0,\ n=4 \\ |f''(t)| \le M \end{array} \right)$

Now $f(t) = t^2+1$, so $f'(t) = 2t$ and $f''(t) = 2$.

$$\Rightarrow |f''(t)| = 2 \le \underbrace{2}_{\text{choose } M=2.}$$

Therefore, $|\text{error}| \le \dfrac{2\cdot(3-0)^3}{24\cdot 4^2} = \dfrac{9}{64} = 0.140625$

* +1 bonus mark ~~s~~ if you explain why it's an underestimate *

3(a) Let $\overbrace{A(a)}^{\text{area}} = \int_a^3 \left(e^{-(s-2)^2} - \frac{1}{e} \right) ds$. (We want to find a which maximizes $A(a)$)

Then, by the FTC part 1,

$$A'(a) = \left(\int_a^3 e^{-(s-2)^2} - \frac{1}{e}\, ds \right)'$$

$$= \left(- \int_3^a e^{-(s-2)^2} - \frac{1}{e}\, ds \right)'$$

Need to switch the Limits of integration to apply the FTC.

$$= -\left(e^{-(a-2)^2} - \frac{1}{e} \right)$$

$$= \frac{1}{e} - e^{-(a-2)^2}$$

Find critical points

So, $\underbrace{A'(a) = 0}$

$$\Rightarrow \frac{1}{e} - e^{-(a-2)^2} = 0$$

$$\Leftrightarrow e^{-1} = e^{-(a-2)^2}$$

$$\Leftrightarrow -1 = -(a-2)^2$$

$$\Leftrightarrow 1 = (a-2)^2$$

$$\Leftrightarrow \pm 1 = (a-2) \Rightarrow \underbrace{a = 1+2 = 3} \text{ or } a = -1+2 = 1.$$

we are told $a < 3$,
so we can disregard this.

Now,

$$A'(0) = \frac{1}{e} - e^{-4} = \frac{1}{e} - \frac{1}{e^4} > 0$$

and

$$A'(2) = \frac{1}{e} - e^{-0} = \frac{1}{e} - 1 < 0$$

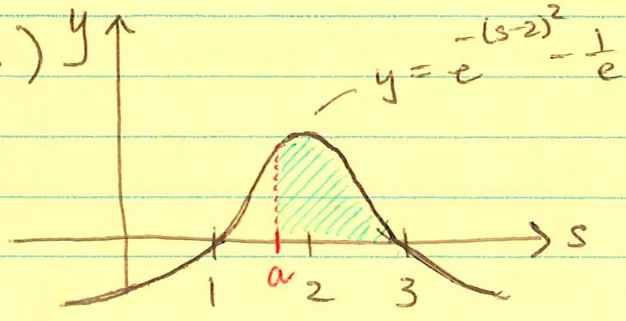
So

	$a=0$	$a=1$	$a=2$
$A'(a)$	$+$		$-$
	$\nearrow$		$\searrow$

$\Rightarrow$ $A(a)$ has a local maximum at $a=1$ (first derivative test).

Since $A(a)$ only has one critical point on $(-\infty, 3)$ it must have a <u>global</u> maximum at $a=1$.

(b)



$\int_a^3 \left(e^{-(s-2)^2} - \frac{1}{e}\right)$ is represented by the shaded region shown. If $a < 1$, we've included area below the s-axis, which is negative. If $a > 1$, then we've excluded some (positive) area under the graph between 1 and a. To maximize the area, we should include all of the positive area and exclude all of the negative area. Thus we should choose $a=1$ to maximize the integral.