Midterm Duration: 80 minutes

*This test has **7 questions** on **9 pages**, for a total of 47 points. Question 7 is a **bonus question**.*

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, watches, cell phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Signature: ______________________

Question:	1	2	3	4	5	6	7	Total
Points:	15	5	6	8	6	7	0	47
Score:								

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

   (i) speaking or communicating with other examination candidates, unless otherwise authorized;

   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;

   (iii) purposely viewing the written papers of other examination candidates;

   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided, but show all of your work. Each part is worth 3 marks, but not all parts are of equal difficulty.

15 marks 1. Evaluate the following integrals.

(a) $\int \left(e^{4x} + 5\cos(2x) + 3x^7 + \frac{1}{x^{1/4}}\right)\, dx$

Solution:
$$\frac{e^{4x}}{4} + \frac{5}{2}\sin(2x) + \frac{1}{8}x^8 + \frac{4}{3}x^{\frac{3}{4}} + C$$

(b) $\int_2^5 \frac{e^{1/x}}{x^2} dx.$

Answer:

Solution: Let $u = \frac{1}{x}$. Then $du = \frac{-1}{x^2}dx$, and $u = \frac{1}{2}$ when $x = 2$ and $u = \frac{1}{5}$ when $x = 5$. So,
$$\int_2^5 \frac{e^{1/x}}{x^2} dx = -\int_{\frac{1}{2}}^{\frac{1}{5}} e^u du = -e^u\Big|_{\frac{1}{2}}^{\frac{1}{5}} = e^{\frac{1}{2}} - e^{\frac{1}{5}}.$$

Remark: An alternative (found by some!) is to use $u = e^{1/x}$, $du = \frac{-1}{x^2}e^{1/x}$.

(c) $\int_1^\pi x^3 \ln(x) dx$

Solution: Integrate by parts: Let $u = \ln(x)$ and $dv = x^3 dx$. Then $du = \frac{1}{x}$, $v = \frac{x^4}{4}$, and
$$\int_1^\pi x^3 \ln(x) dx = \left[\frac{x^4}{4}\ln(x)\right]_1^\pi - \int_1^\pi \frac{x^4}{4}\cdot\frac{1}{x}dx = \left[\frac{x^4}{4}\ln(x)\right]_1^\pi - \int_1^\pi \frac{x^3}{4}dx$$
$$= \left[\frac{\pi^4}{4}\ln(\pi) - 0\right] - \left[\frac{\pi^4}{16} - \frac{1}{16}\right]$$
$$= \frac{\pi^4\ln(\pi) - \pi^4 + 1}{16}.$$

(d) $\displaystyle\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}dx$

Answer:

Solution: First, let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}$, and

$$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}dx = 2\int \sin^3(u)du.$$

Since we have an **odd** power of $\sin(x)$, we use the trig identity $\sin^2(u)+\cos^2(u) = 1$:

$$2\int \sin^3(u)du = 2\int \sin^2(u)\sin(u)du = 2\int (1-\cos^2(u))\sin(u)du.$$

Now let $v = \cos(u)$. Then $dv = -\sin(u)du$ and we have that

$$\begin{aligned}\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}dx = -2\int (1-v^2)dv = 2\int (v^2-1)dv &= \frac{2}{3}v^3 - 2v + C\\ &= \frac{2}{3}\cos^3(u) - 2\cos(u) + C\\ &= \frac{2}{3}\cos^3(\sqrt{x}) - 2\cos(\sqrt{x}) + C.\end{aligned}$$

(e) *Without* calculating the resulting coefficients, and with no reference to a specific integral, give the appropriate form of the partial fraction decomposition that will decompose the following fractions into a sum of simpler fractions.

(i) $\displaystyle\frac{15}{(x+3)(x-1)}$

(ii) $\displaystyle\frac{5x}{(x-1)(x^2+x+1)}$

(iii) $\displaystyle\frac{3x+1}{(x+1)(x^2+2x-3)}$

Answer:

Solution: (i) $\displaystyle\frac{A}{x+3} + \frac{B}{x-1}$.

(ii) $\displaystyle\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$.

(Note: The discriminant is $\Delta = 1^2 - 4\cdot 1\cdot 1 = -3 < 0$.)

(iii) $\displaystyle\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+3}$.

(Note: $x^2+2x-3 = (x-1)(x+3)$.)

3 marks 2. (a) Let $G(x) = \int_1^{x^2} e^{e^t}\, dt$. Calculate $G'(1)$.

Solution: Let $F(x) = \int_1^x e^{e^t}\, dt$. Then $G(x) = F(x^2)$ and, by the Chain Rule,

$$G'(x) = F'(x^2) \cdot 2x.$$

Also, since e^{e^t} is continuous, by the Fundamental Theorem of Calculus part 1 we know that

$$F'(x) = e^{e^x}.$$

Therefore

$$G'(x) = F'(x^2) \cdot 2x = e^{e^{x^2}} \cdot 2x,$$

and so

$$G'(1) = 2e^e.$$

2 marks (b) Find the equation of the line tangent to $G(x)$ at the point $x = 1$.

Solution: From (a) we know that the slope of the line tangent to $G(x)$ at $x = 1$ is $m = 2e^e$. Using the point-slope formula for a line with the point $(1, G(1))$, we have that the equation of the tangent line is

$$y - G(1) = 2e^e(x - 1).$$

Now,

$$G(1) = \int_1^1 e^{e^t}\, dt = 0$$

since there is no area below the graph between $x = 1$ and $x = 1$. So, the equation of the tangent line is

$$y = 2e^e(x - 1).$$

Long-Solution Problems. In questions 3–7, justify your answers and **show all of your work**. Unless otherwise indicated, **simplification of answers is not required in these questions**.

3 marks 3. (a) Approximate $\int_1^2 \ln(x)dx$ using a *midpoint* Riemann sum with $n = 3$.

Solution: Here $a = 1$ and $b = 2$, so $\Delta x = \frac{b-a}{n} = \frac{2-1}{3} = \frac{1}{3}$. Furthermore, we have that $f(x) = \ln(x)$ and, since we're asked to use midpoints, $x_k^* = a + (k - \frac{1}{2})\Delta x = 1 + (k - \frac{1}{2})\frac{1}{3}$.

So

$$\int_1^2 \ln(x)dx \approx M_3 = \sum_{k=1}^{3} f(x_k^*)\Delta x = \frac{1}{3}\left(\ln\left(\frac{7}{6}\right) + \ln\left(\frac{9}{6}\right) + \ln\left(\frac{11}{6}\right)\right).$$

2 marks (b) Use the formula $E_M \leq \dfrac{M(b-a)^3}{24n^2}$ to find a bound for the absolute error in the approximation in (a).

Solution: By definition, M is a number such that $|f''(x)| \leq M$ for all $x \in [1, 2]$. Since $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$ and $f''(x) = \frac{-1}{x^2}$. Therefore $|f''(x)| = \frac{1}{x^2}$. Now, since $\frac{1}{x^2}$ is a decreasing function, we have that $|f''(x)| \leq \frac{1}{1^2} = 1$, so we can take $M = 1$.

Thus,

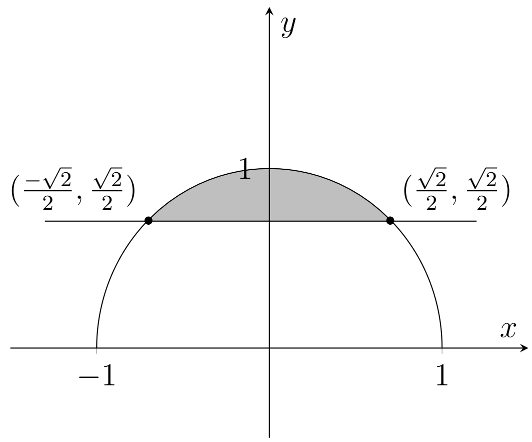
$$|\text{error}| \leq 1 \cdot \frac{(2-1)^3}{24 \cdot 3^2} = \frac{1}{24 \cdot 3^2}.$$

1 mark (c) Given that $\int_1^2 \ln(x)dx = 2\ln(2) - 1$, what is the true error in the approximation in (a). You do **not** need to evaluate your answer.

Solution: The true error is given by

$$\left|\left(\int_1^2 \ln(x)dx\right) - M_3\right| = \left|(2\ln(2) - 1) - \frac{1}{3}\left(\ln\left(\frac{7}{6}\right) + \ln\left(\frac{9}{6}\right) + \ln\left(\frac{11}{6}\right)\right)\right|.$$

8 marks 4. Use a method of integration to calculate the area of the cap of the circle $x^2 + y^2 = 1$ shown below. If you are unable to determine the appropriate limits of integration, calculate the corresponding indefinite integral.



Solution: The equation of the top-half of the circle is $y = \sqrt{1-x^2}$. Now, the area of the cap of the circle is equal to the total area between the top-half of the circle and the x-axis from $x = -\frac{\sqrt{2}}{2}$ to $x = \frac{\sqrt{2}}{2}$ (let's call this area A_1) minus the area of the rectangle directly below the cap (let's call this area A_2). In the language of integrals,

$$A_1 = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2}dx \qquad \text{and} \qquad A_2 = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{\sqrt{2}}{2}dx,$$

since we can think of the rectangle as the area under the graph $y = \frac{\sqrt{2}}{2}$ from $x = -\frac{\sqrt{2}}{2}$ to $x = \frac{\sqrt{2}}{2}$. To calculate A_1, we make the trigonometric substitution $x = \sin(\theta)$. Then $dx = \cos(\theta)d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta)$. Also, when $x = -\frac{\sqrt{2}}{2}$ we have that $-\frac{\sqrt{2}}{2} = \sin(\theta)$, so $\theta = -\frac{\pi}{4}$. Similarly, when $x = \frac{\sqrt{2}}{2}$, $\theta = \frac{\pi}{4}$. Thus we have that

$$A_1 = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(\theta)d\theta.$$

To calculate this integral, we need to make use of the double-angle formula: $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$. This gives us

$$\begin{aligned} A_1 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\cos(2\theta)d\theta &= \left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\right) \\ &= \frac{\pi}{8} + \frac{1}{4}\sin\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{8} + \frac{1}{4}\sin\left(-\frac{\pi}{2}\right)\right) \\ &= \frac{\pi}{4} + \frac{1}{2}. \end{aligned}$$

Now,

$$A_2 = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{\sqrt{2}}{2}dx = \frac{\sqrt{2}}{2}x\Big|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} = 1.$$

Therefore the area of the cap is $\frac{\pi}{4} + \frac{1}{2} - 1 = \frac{\pi}{4} - \frac{1}{2}$.

5. The *average value* of an integrable function $f(t)$ over the interval $[a, b]$ is defined to be

$$f_{ave} = \frac{1}{b-a}\int_a^b f(t)dt.$$

5 marks (a) If the velocity of a particle after t seconds is given by $v(t) = te^{-t}$, find the particle's average velocity over the first 2 seconds.

Solution: We need to calculate $v_{ave} = \frac{1}{2-0}\int_0^2 te^{-t}dt.$

Let's first calculate $I = \int_0^2 te^{-t}dt$. We will use the method of integration by parts with $u = t$ and $dv = e^{-t}dt$. Then $du = 1dt$ and $v = -e^{-t}$.

This gives us that

$$\begin{aligned} I = \int_0^2 te^{-t}dt &= \Big[t(-e^{-t})\Big]_0^2 - \int_0^2 (-e^{-t})dt \\ &= \Big[t(-e^{-t})\Big]_0^2 + \int_0^2 e^{-t}dt \\ &= \Big[t(-e^{-t})\Big]_0^2 - e^{-t}\Big|_0^2 \\ &= (-2e^{-2} - 0) - (e^{-2} - e^0) \\ &= 1 - 3e^{-2}. \end{aligned}$$

So,

$$v_{ave} = \frac{1-3e^{-2}}{2} \quad \text{units of length/seconds.}$$

Remark: Many people tried to use IBP with

$$u = e^{-t},\ dv = tdt \Rightarrow du = -e^{-t}dt,\ v = \frac{t^2}{2}.$$

This results in the expression (ignoring the limits of integration for brevity)

$$e^{-t}\cdot\frac{t^2}{2} - \int (-e^{-t})\frac{t^2}{2}dt$$

and the integral that appears is <u>much harder</u> than the original one.

1 mark (b) What is the particle's displacement after 2 seconds have passed?

Solution:

$$x(2) - x(0) = \int_0^2 v(t)dt = \int_0^2 te^{-t}dt = 1 - 3e^{-2} \quad \text{units of length.}$$

6. The population dynamics of a population of fish $x(t)$ (in units of millions of fish) at time t can be modelled by the differential equation

$$\frac{dx}{dt} = 0.2(x - x^2).$$

3 marks (a) Find the partial fraction decomposition of

$$\frac{1}{x(1-x)}.$$

Solution: The partial fraction decomposition has the form

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x},$$

which tells us that

$$1 = A(1-x) + Bx = (B-A)x + A.$$

So, $A = 1$ and $B - A = 0 \Rightarrow B = 1$. Therefore,

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}.$$

3 marks (b) Find the general solution of the differential equation above. You may leave your answer in implicit form.

Solution: Moving all the x-terms to the left and all of the t-terms to the right we get

$$\frac{1}{x-x^2}dx = 0.2dt.$$

Integrating on both sides (and noting that $x - x^2 = x(1-x)$) we have

$$\begin{aligned} \int \frac{1}{x(1-x)}dx &= \int 0.2dt \\ \Rightarrow \int \frac{1}{x} + \frac{1}{1-x}dx &= 0.2t + C \\ \Rightarrow \ln|x| - \ln|1-x| &= 0.2t + C. \end{aligned}$$

1 mark (c) If, initially, there are 1.5 million fish, will the population of fish initially increase or decrease? Justify your answer. (Hint: You do not need to solve (b) to answer this question.)

Solution: The rate of change of the population is given by $\frac{dx}{dt} = 0.2(x - x^2) = 0.2x(1-x)$. If the initial population is $x(0) = 1.5$, then $\frac{dx}{dt} = 0.2 \cdot (1.5)(1 - 1.5) < 0$, which indicates that the population is initially decreasing.

7. **Bonus [2 marks]** A function $f(x)$ is said to be *even* if $f(x) = f(-x)$ for all x. For example, $f(x) = |x|$ is an even function.

   If $f(x)$ is an even integrable function, show that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

   for any constant $a > 0$.

Solution: For any $a > 0$, we have that

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$$

Now, for the integral $\int_{-a}^{0} f(x)dx$, let $u = -x$. Then $du = -dx$ and when $x = -a$, $u = -(-a) = a$. Similarly, when $x = 0$, $u = 0$. Thus,

$$\int_{-a}^{0} f(x)dx = \int_{a}^{0} -f(-u)du = \int_{0}^{a} f(-u)du.$$

Since f is an even function, $f(-u) = f(u)$ for all u. Hence

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(u)du = \int_{0}^{a} f(x)dx$$

(since u is simply a dummy variable, we can change it to x in the last step). Therefore

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$