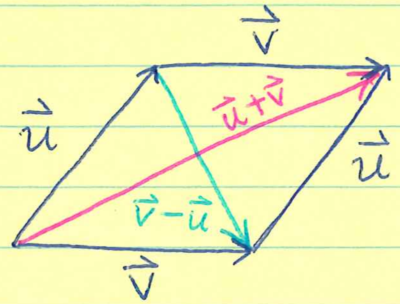
Homework #5 — Solutions.

① Let $\vec{u}$ and $\vec{v}$ represent the sides of the parallelogram (as shown):



Then one diagonal is given by $\vec{u}+\vec{v}$ and the other is given by $\vec{v}-\vec{u}$.

We need to show that $\vec{u}+\vec{v}$ is perpendicular to $\vec{v}-\vec{u}$, so it suffices to show that

$$(\vec{u}+\vec{v})\cdot(\vec{v}-\vec{u}) = 0.$$

Now,

$$(\vec{u}+\vec{v})\cdot(\vec{v}-\vec{u}) = \cancel{\vec{u}\cdot\vec{v}} + \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{u} - \cancel{\vec{v}\cdot\vec{u}}$$
$$= \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{u}$$

Also, $\vec{v}\cdot\vec{v} = \|\vec{v}\|\cdot\|\vec{v}\|\cos(0)$ angle between $\vec{v}$ and $\vec{v}$ is 0.

$$= \|\vec{v}\|^2.$$

and, similarly, $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$.

So, since we're told that $\|\vec{u}\| = \|\vec{v}\|$, we get

$$(\vec{u}+\vec{v})\cdot(\vec{v}-\vec{u}) = \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{u}$$
$$= \|\vec{v}\|^2 - \|\vec{u}\|^2 = 0.$$

$\Rightarrow$ $\vec{u}+\vec{v}$ and $\vec{v}-\vec{u}$ are perpendicular.

2 First note that $f(-1,1) = \int_{(-1)^2}^{1} e^{-t^2}\,dt = \int_{1}^{1} e^{-t^2}\,dt = 0$.

Now, by the FTC.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(-\int_{y}^{x^2} e^{-t^2}\,dt\right) = -e^{-x^4}\cdot 2x$$

so

$$\left.\frac{\partial f}{\partial x}\right|_{(-1,1)} = -e^{-(-1)^4}\cdot 2(-1) = 2e^{-1} = \frac{2}{e}.$$

Also,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int_{x^2}^{y} e^{-t^2}\,dt\right) = e^{-y^2}$$

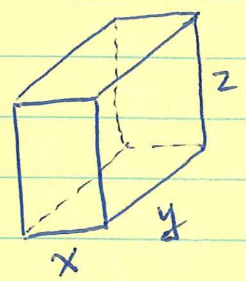
so

$$\left.\frac{\partial f}{\partial y}\right|_{(-1,1)} = e^{-(1)^2} = e^{-1} = \frac{1}{e}.$$

$\therefore$ The equation of the tangent plane is.

$$z - 0 = \frac{2}{e}(x+1) + \frac{1}{e}(y-1).$$

3 [Bonus] Let x, y, z be the dimensions of the box (as shown).



Then $SA = 2xy + 2xz + 2yz$.

We are also given that $V = 1000$,

so $x \cdot y \cdot z = 1000$

$$\Rightarrow z = \frac{1000}{xy}.$$

$$\therefore SA = 2xy + 2\cancel{x}\left(\frac{1000}{\cancel{x}y}\right) + 2\cancel{y}\left(\frac{1000}{x\cancel{y}}\right).$$

(note: $x, y \neq 0$ since otherwise $V \neq 1000$).

$$= 2xy + \frac{2000}{y} + \frac{2000}{x} = \underbrace{f(x,y)}_{\text{define } f(x,y) \text{ this way.}}$$

Then $\dfrac{\partial f}{\partial x} = 2y - \dfrac{2000}{x^2} = 0 \Rightarrow y = \dfrac{1000}{x^2}$.

$$\frac{\partial f}{\partial y} = 2x - \frac{2000}{y^2} = 0 \Rightarrow x = \frac{1000}{y^2}.$$

$$\Rightarrow x = \frac{1000}{\left(\frac{1000}{x^2}\right)^2} = \frac{1000}{1000^2} \cdot x^4 = \frac{x^4}{1000}$$

$$\Rightarrow x = \frac{x^4}{1000}$$

Again since $x \neq 0$, this gives

$$1000 = x^3 \Rightarrow x = 10. \text{ (cm)}$$

$$\Rightarrow y = \frac{1000}{x^2} = \frac{1000}{100} = 10 \text{ (cm)}.$$

So, the only critical point is $(x,y) = (10,10)$.

Now,

$$\frac{\partial^2 f}{\partial x^2} = \frac{4000}{x^3}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{4000}{y^3} \text{ and } \frac{\partial^2 f}{\partial y \partial x} = 2.$$

So

CP (x_0, y_0)	$f_{xx}(x_0,y_0)$	$f_{yy}(x_0,y_0)$	$f_{xy}(x_0,y_0)$	D
$(10,10)$	4	4	2	$4^2 - 2^2 = 12 > 0$
	>0			

So the 2nd Derivatives Test tells us that there is a local minimum at $(10,10)$.

Intuitively, there must be a global minimum, so this must be it!

$$\therefore z = \frac{1000}{xy} = \frac{1000}{10 \cdot 10} = 10. \text{ (cm)}$$

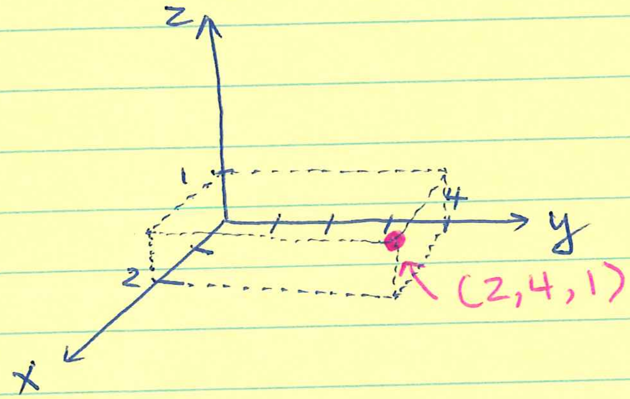
$\Rightarrow$ The box of volume 1000 cm^3 with minimal surface area has dimensions

$$x = 10 \text{ cm}, \quad y = 10 \text{ cm}, \quad z = 10 \text{ cm} \qquad \text{(cube)}.$$

Lecture Examples – August 3rd

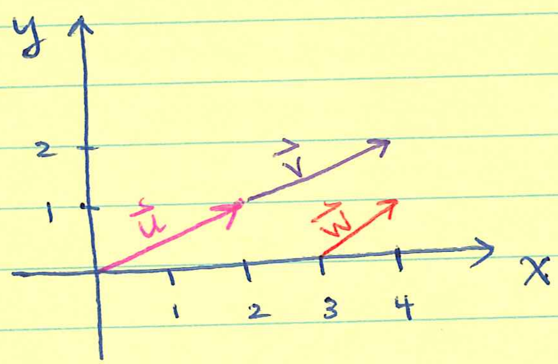
3-D Coordinate System

Example: Plot the point $(2, 4, 1)$.



Vectors

Example: Find the components of the vectors shown.



$\vec{u} = \langle 2, 1 \rangle$
$\vec{v} = \langle 4-2, 2-1 \rangle = \langle 2, 1 \rangle$ } same!
$\vec{w} = \langle 4-3, 1-0 \rangle = \langle 1, 1 \rangle$

Example: Find the length of the vector $\langle 1, 2, 1 \rangle$.

$$\| \langle 1, 2, 1 \rangle \| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

Example: If $\vec{u} = \langle 1,1,4 \rangle$ and $\vec{v} = \langle 2,0,-3 \rangle$, find $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

$$\vec{u}+\vec{v} = \langle 1,1,4 \rangle + \langle 2,0,-3 \rangle = \underbrace{\langle 1+2, 1+0, 4-3 \rangle}_{\text{add component-wise.}} = \langle 3,1,1 \rangle.$$

$$\vec{u}-\vec{v} = \langle 1,1,4 \rangle - \langle 2,0,-3 \rangle = \langle 1-2, 1-0, 4--3 \rangle = \langle -1,1,7 \rangle$$

Example: If $\vec{u} = \langle 1,1,4 \rangle$, and $\vec{v} = \langle 2,0,-3 \rangle$, find $5\vec{u}$, $-\frac{1}{2}\vec{u}$ and $0\vec{v}$.

$$5\vec{u} = 5\langle 1,1,4 \rangle = \underbrace{\langle 5\cdot 1, 5\cdot 1, 5\cdot 4 \rangle}_{\text{multiply component-wise.}} = \langle 5,5,20 \rangle$$

$$-\frac{1}{2}\vec{u} = -\frac{1}{2}\langle 1,1,4 \rangle = \langle -\frac{1}{2}\cdot 1, -\frac{1}{2}\cdot 1, -\frac{1}{2}\cdot 4 \rangle = \langle -\frac{1}{2}, -\frac{1}{2}, -2 \rangle.$$

$$0\cdot\vec{v} = 0\langle 2,0,-3 \rangle = \langle 0\cdot 2, 0\cdot 0, 0\cdot(-3) \rangle = \langle 0,0,0 \rangle$$

Example: If $\vec{u} = \langle 1,1,2 \rangle$, $\vec{v} = \langle 3,0,-1 \rangle$, $\vec{w} = \langle 2,2,3 \rangle$, calculate the following.

$$\vec{u}\cdot\vec{v} = \langle 1,1,2 \rangle \cdot \langle 3,0,-1 \rangle = 1\cdot 3 + 1\cdot 0 + 2\cdot(-1) = 3-2 = 1$$

$$\vec{u}\cdot\vec{w} = \langle 1,1,2 \rangle \cdot \langle 2,2,3 \rangle = 1\cdot 2 + 1\cdot 2 + 2\cdot 3 = 2+2+6 = 10$$

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} = 1+10 = 11.$$

$$5(\vec{u}\cdot\vec{v}) = (5\vec{u})\cdot\vec{v} = \vec{u}\cdot(5\vec{v}) = 5\cdot 1 = 5.$$

Functions of 2 Variables

Example: If $f(x,y) = x^2y + e^{x+y}$, find $f(0,0)$, $f(1,0)$ and $f(1,2)$.

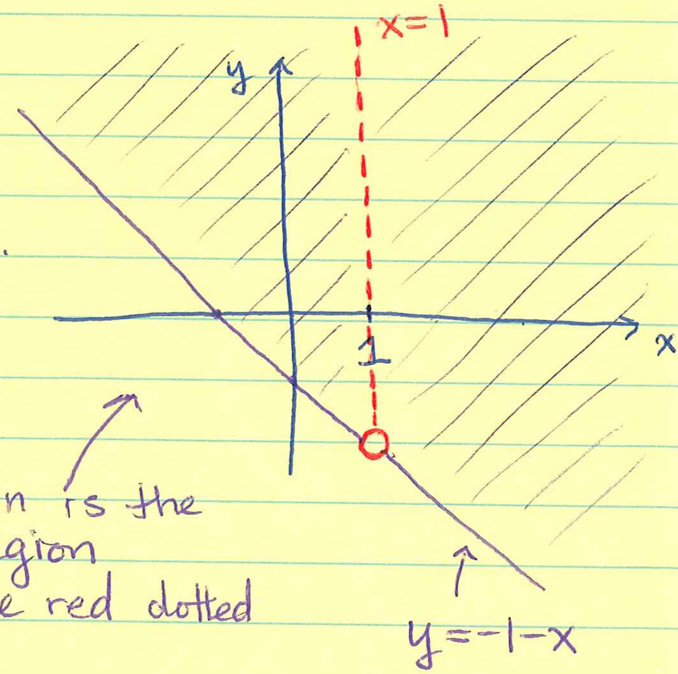
$$f(0,0) = 0^2 \cdot 0 + e^{0+0} = e^0 = 1.$$

$$f(1,0) = 1^2 \cdot 0 + e^{1+0} = e^1 = e.$$

$$f(1,2) = 1^2 \cdot 2 + e^{1+2} = 2 + e^3$$

Example: Find the domains of the following functions.

① $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$



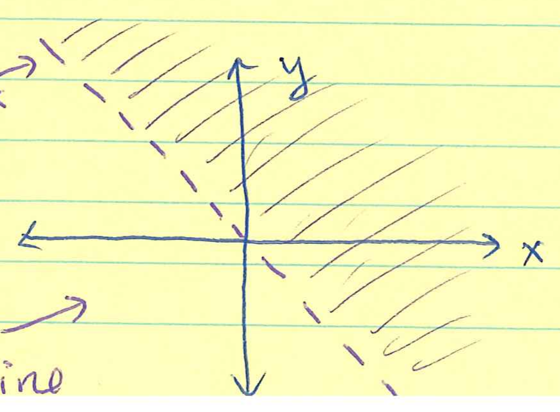
Need $x+y+1 \geq 0$ and $x-1 \neq 0$.

$\Leftrightarrow$ $\underbrace{y \geq -1-x}_{\text{focus on } y=-1-x}$ and $x \neq 1$.

the domain is the shaded region (excluding the red dotted line)

$y=-1-x$

② $f(x,y) = x \cdot \ln(x+y)$



$y=-x$

Need $x+y>0 \Leftrightarrow y>-x$

domain excludes dotted line

3 $f(x,y) = \sqrt{16 - x^2 - y^2}$

Need $16 - x^2 - y^2 \geq 0$

$\iff \underbrace{x^2 + y^2 \leq 16}$

focus on $x^2 + y^2 = 16 \rightarrow$ circle of radius 4