Lecture Examples — Thursday, July 20th

Probability:

If Luiz's chalk throwing has pdf

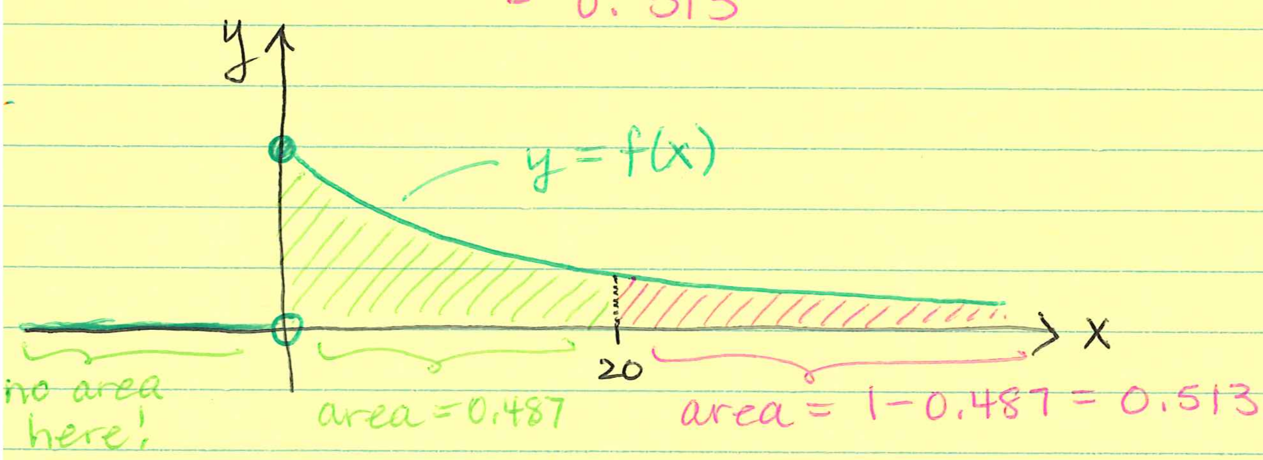
$$f(x) = \begin{cases} \frac{1}{30} e^{-x/30} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

① What is the probability Luiz's throw is further than 20cm away from the target?

A: $\mathbb{P}(X > 20) = 1 - \mathbb{P}(X \leq 20)$ $\left(\underbrace{\mathbb{P}(X \leq 20) + \mathbb{P}(X > 20)}_{\text{total probability.}} = 1 \right)$.

$= 1 - \underbrace{0.487}_{\text{from Monday.}}$

$= 0.513$



② What is the probability Luiz's throw is exactly 20cm away from the target?

A:
$\mathbb{P}(X = 20) = \mathbb{P}(20 \leq X \leq 20) = \underbrace{\int_{20}^{20} f(x)\,dx}_{\text{no area between 20 and 20!}} = 0$

3 What is the expected value of the distance between Luiz's chalk mark and the target.

A:

$$\overbrace{\mathbb{E}(X) = \int_{-\infty}^{\infty} x\cdot f(x)\,dx}^{\text{by definition}} = \cancelto{0}{\int_{-\infty}^{0} x\cdot f(x)\,dx} + \int_{0}^{\infty} x\cdot f(x)\,dx$$

$f(x)=0$ for $x\in(-\infty,0)$

$$= \int_0^\infty x\cdot \frac{1}{30}e^{-x/30}\,dx$$

IBP:

$u = x \qquad dv = \frac{1}{30}e^{-x/30}\,dx$

$du = dx \qquad v = -e^{-x/30}$

$$= -xe^{-x/30}\Big|_0^\infty + \int_0^\infty e^{-x/30}\,dx$$

"plug in" ∞

$$= \Big(\lim_{x\to\infty} \underbrace{-xe^{-x/30}}_{\substack{-\infty\cdot 0 \\ \text{indeterminate form}}}\Big) - 0 + \Big(-30e^{-x/30}\Big|_0^\infty\Big).$$

"plug in" ∞

$$= \Big(\lim_{x\to\infty} \underbrace{\frac{-x}{e^{x/30}}}_{\substack{\frac{-\infty}{\infty} \\ \text{Appropriate indeterminate form for applying L'Hôpital's Rule}}}\Big) + \underbrace{\Big(\lim_{x\to\infty} -30e^{-x/30}\Big)}_{=-30\cdot 0 = 0} - \Big(-30\underbrace{e^{0}}_{=1}\Big)$$

$\overset{L'H}{=} \left(\lim_{x\to\infty} \frac{\overset{=(-x)'}{-1}}{\underbrace{\frac{1}{30} e^{x/30}}_{=(e^{x/30})'}} \right) + 30$ $\left(\lim_{x\to\infty} \frac{1}{30} e^{x/30} = \infty \right)$

$= 0 + 30 = 30$ cm //

④ Calculate the variance of X.

A: The variance is:

$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \mathbb{E}(X^2) - [\overset{\text{from ③}}{30}]^2$

$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \cancel{\int_{-\infty}^{0} x^2 \cdot f(x)\,dx}^{\,0} + \int_0^{\infty} x^2 \cdot f(x)\,dx$

$= \int_0^{\infty} x^2 \cdot \frac{1}{30} e^{-x/30}\,dx$ IBP: $u = x^2$, $dv = \frac{1}{30} e^{-x/30}\,dx$, $du = 2x\,dx$, $v = -e^{-x/30}$

$= -x^2 e^{-x/30} \Big|_0^{\infty} + \int_0^{\infty} +2x e^{-x/30}\,dx$

$= \left(\lim_{x\to\infty} \underbrace{-x^2 e^{-x/30}}_{-\infty \cdot 0} \right) - 0 + 2\int_0^{\infty} x e^{-x/30}\,dx$

indeterminate form

From ③, we know that

$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} x \cdot \frac{1}{30} e^{-x/30}\,dx = 30$

$\therefore \int_0^{\infty} x e^{-x/30}\,dx = 30 \int_0^{\infty} x \cdot \frac{1}{30} e^{-x/30}\,dx = 30 \cdot 30$

So

$$\mathbb{E}(X^2) = \left(\underbrace{\lim_{x\to\infty} \frac{-x^2}{e^{x/30}}}_{-\frac{\infty}{\infty} \to \text{ indeterminate form appropriate for applying L'Hôpital's Rule.}} \right) + 2\cdot 30\cdot 30$$

$\downarrow$ L'H

$$= \left(\lim_{x\to\infty} \frac{-2x}{\underbrace{\frac{1}{30}e^{x/30}}} \right) + 1800$$

$-\frac{\infty}{\infty} \to$ apply L'Hôpital's Rule again.

$$= \left(\lim_{x\to\infty} \frac{-2}{\frac{1}{900}e^{x/30}} \right) + 1800 \qquad \left(\lim_{x\to\infty} \frac{1}{900}e^{x/30} = \infty \right)$$

$$= 1800.$$

$\therefore \operatorname{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 1800 - 900 = 900$ ///.

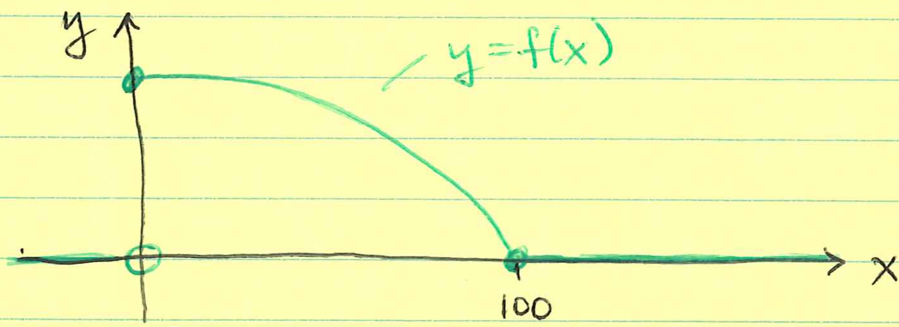
⑤ Calculate the standard deviation in X

A: The standard deviation is:

$$\sigma(X) = \sqrt{\operatorname{Var}(X)} = \sqrt{900} = 30 \text{ cm}.$$

Example: Given the pdf for chalk throwing

$$f(x) = \begin{cases} \frac{3}{200}\left(1 - \frac{x^2}{10{,}000}\right) & 0 \le x \le 100 \\ 0 & \text{otherwise.} \end{cases}$$



① Calculate the probability that the chalk hits no further than 10 cm away from the target.

A: The probability is

$$\mathbb{P}(X \le 10) = \mathbb{P}(-\infty \le X \le 10) = \int_{-\infty}^{10} f(x)\,dx$$

$$= \int_{-\infty}^{0} \cancelto{0}{f(x)}\,dx + \int_{0}^{10} f(x)\,dx$$

$$= \int_{0}^{10} \frac{3}{200}\left(1 - \frac{x^2}{10{,}000}\right) dx$$

$$= \frac{3}{200}\left[\, x - \frac{x^3}{30{,}000} \Big|_0^{10} \right]$$

$$= \frac{3}{200}\left[10 - \frac{10^3}{30{,}000} - 0\right] = \frac{3 \cdot 299}{6000} = 0.1495$$

(about 15% chance)

2. Find the expected distance between the chalk mark and the target.

A: The expected distance is the expected value of X, and

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \cancelto{0}{\int_{-\infty}^{0} x \cdot f(x)\,dx} + \int_{0}^{100} x \cdot f(x)\,dx + \cancelto{0}{\int_{100}^{\infty} x \cdot f(x)\,dx}$$

$$= \int_{0}^{100} x \cdot \frac{3}{200}\left(1 - \frac{x^2}{10{,}000}\right) dx$$

$$= \frac{3}{200} \int_{0}^{100} \left(x - \frac{x^3}{10{,}000}\right) dx$$

$$= \frac{3}{200}\left[\frac{1}{2}x^2 - \frac{x^4}{40{,}000} \Bigg|_{0}^{100}\right]$$

$$= \frac{3}{200}\left[\frac{1}{2}(100)^2 - \frac{(100)^4}{40{,}000} - 0\right] = 37.5 \text{ cm}$$

3. Find the variance in X.

A: The variance of X is

$$Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

We already know (from 2) that $[\mathbb{E}(X)]^2 = [37.5]^2$.

Now,

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_{-\infty}^{0} x^2 \cdot f(x)\,dx + \int_{0}^{100} x^2 \cdot f(x)\,dx + \int_{100}^{\infty} x^2 \cdot f(x)\,dx$$

$$= \int_0^{100} x^2 \cdot \frac{3}{200}\left(1 - \frac{x^2}{10{,}000}\right) dx$$

$$= \frac{3}{200} \int_0^{100} \left(x^2 - \frac{x^4}{10{,}000}\right) dx$$

$$= \frac{3}{200}\left(\frac{1}{3}x^3 - \frac{1}{50{,}000}x^5 \Big|_0^{100}\right)$$

$$= \frac{3}{200}\left(\frac{1}{3}(100)^3 - \frac{1}{50{,}000}(100)^5 - 0\right)$$

$$= 2000$$

$\therefore$ $Var(X) = 2000 - (37.5)^2 = 593.75$

④ Find the Standard deviation in X.

<u>A:</u> We have that the standard deviation is.

$$\sigma(X) = \sqrt{Var(X)} = \sqrt{593.75} \approx 24.37 \text{ cm.}$$