Lecture Examples – August 4th

Level Curves

① Example: Plot some level curves of the following functions

(a) $f(x,y)=\sqrt{16-x^2-y^2}$

Set $z=k \Rightarrow k=\sqrt{16-x^2-y^2}$
$\Rightarrow k^2=16-x^2-y^2$
$\Rightarrow x^2+y^2=16-k^2$ } circle of radius $\sqrt{16-k^2}$

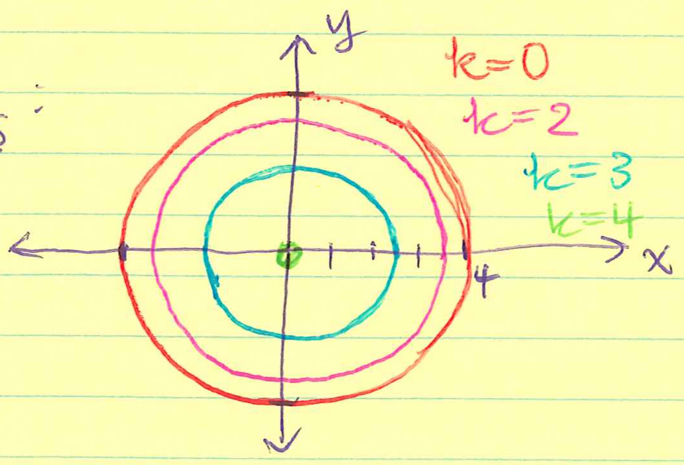
$\underline{k=0}$: $x^2+y^2=16$ } circle of radius 4.

$\underline{k=2}$: $x^2+y^2=16-4=12$ } circle of radius $\sqrt{12}\approx 3.5$

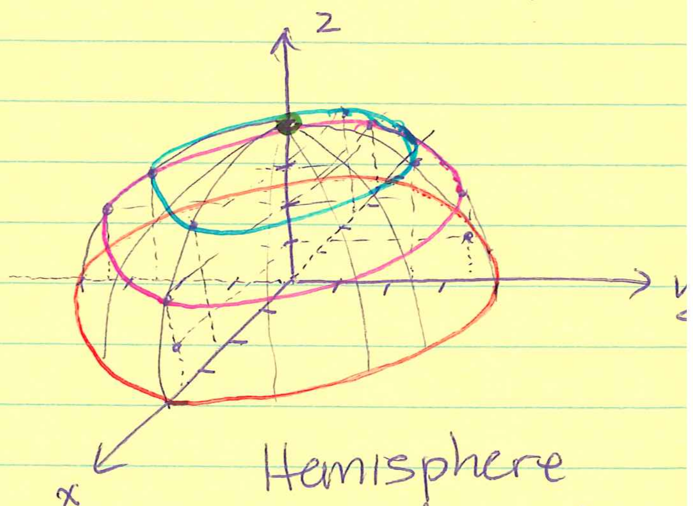
$\underline{k=3}$: $x^2+y^2=16-9=7$ } circle of radius $\sqrt{7}\approx 2.6$.

$\underline{k=4}$: $x^2+y^2=16-16=0$ } point ~~circle~~ $x=0, y=0$.

Level curves:



3-D (not on exam):



Hemisphere of radius 4.

b) $f(x, y) = 2x^2 + 2y^2$

Set $z = k \Rightarrow k = 2x^2 + 2y^2$

$\Rightarrow \frac{k}{2} = x^2 + y^2$ } circle of radius $\sqrt{\frac{k}{2}}$

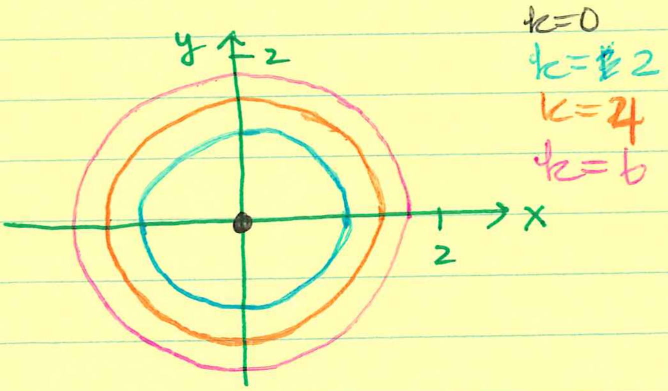
$\underline{k=0:}$ $x^2 + y^2 = 0$ } point $x=0, y=0$.

$\underline{k=2:}$ $x^2 + y^2 = 1$ } circle of radius 1

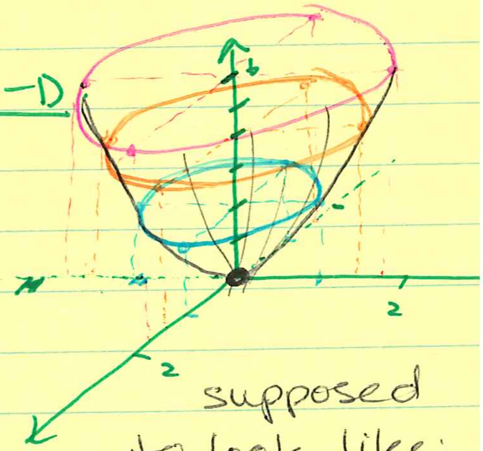
$\underline{k=4:}$ $x^2 + y^2 = 2$ } circle of radius $\sqrt{2} \approx 1.4$

$\underline{k=6:}$ $x^2 + y^2 = 3$ } circle of radius $\sqrt{3} \approx 1.7$

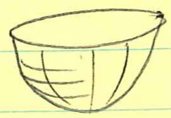
<u>Level curves</u>:



<u>3-D</u>



supposed to look like:



<u>Partial Derivatives</u>

<u>Example</u>: If $f(x, y) = 2e^{xy} + x \cdot \sin(\pi x + y) + \overset{2}{\cancel{2}}$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\left.\frac{\partial f}{\partial x}\right|_{(0,0)}$ and $\left.\frac{\partial f}{\partial y}\right|_{(1,0)}$.

$$\frac{\partial f}{\partial x} = 2e^{xy}\cdot y + 1\cdot \sin(\pi x+y) + x\cdot\cos(\pi x+y)\cdot\pi + 0$$

$$\frac{\partial f}{\partial y} = 2e^{xy}\cdot x + x\cdot\cos(\pi x+y)\cdot 1 + 0$$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 2e^{0}\cdot 0 + \sin(0+0) + 0\cdot\cos(0+0)\cdot\pi = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,0)} = 2e^{0}\cdot 1 + 1\cdot\cos(\pi\cdot 1+0)\cdot 1 = 2-1 = 1.$$

Example: If $f(x,y) = e^{x+y} + 2xy + y^2$, find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial y\,\partial x}$ and $\frac{\partial^2 f}{\partial x\,\partial y}$.

First, we have to calculate the first partial derivatives.

$$\frac{\partial f}{\partial x} = e^{x+y} + 2y + 0, \quad \frac{\partial f}{\partial y} = e^{x+y} + 2x + 2y$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = e^{x+y} + 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = e^{x+y} + 0 + 2.$$

Also

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial y\,\partial x} &= \frac{\partial}{\partial y}\left(e^{x+y} + 2y\right) = e^{x+y} + 2 \\ \frac{\partial^2 f}{\partial x\,\partial y} &= \frac{\partial}{\partial x}\left(e^{x+y} + 2x + 2y\right) = e^{x+y} + 2 \end{aligned}\right\} \text{SAME!}$$

Directional Derivatives

Example: If $f(x,y) = e^{x+y} + 2xy + y^2$ and $\vec{u} = \langle \frac{1}{\sqrt{2}},$

find $D_{\vec{u}} f(1,0)$.

To calculate directional derivatives, we need to make sure our direction vector is a unit vector.

Check: $\| \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle \| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} =$

(If it's not a unit vector, first find the corresponding unit vector and use that as $\vec{u}$

From the previous example, we had.

$$\frac{\partial f}{\partial x} = e^{x+y} + 2y \quad \text{and} \quad \frac{\partial f}{\partial y} = e^{x+y} + 2x + 2y.$$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,0)} = e^{1+0} + 2 \cdot 0 = e, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,0)} = e^{1+0} + 2 \cdot 1 + 2 \cdot 0 = e+$$

$$\therefore D_{\vec{u}} f(1,0) = \left(\left.\frac{\partial f}{\partial x}\right|_{(1,0)} \right) \cdot \left(\frac{1}{\sqrt{2}} \right) + \left(\left.\frac{\partial f}{\partial y}\right|_{(1,0)} \right) \cdot \left(\frac{-1}{\sqrt{2}} \right)$$

$$= \frac{e}{\sqrt{2}} + \frac{-(e+2)}{\sqrt{2}} = \frac{e-e-2}{\sqrt{2}} = \frac{-2}{\sqrt{2}}$$

$$= -\sqrt{2}$$

Gradients.

Example: If $f(x,y) = \sqrt{9-x^2-y^2}$, find the magnitude and a vector pointing in the direction of steepest slope at $(2,2)$.

We know that the magnitude of the steepest slope at $(2,2)$ is $\|\vec{\nabla} f(2,2)\|$ and that the gradient vector itself $(\vec{\nabla} f(2,2))$ points in the direction of the steepest slope.

$$\frac{\partial f}{\partial x} = \frac{1}{2}(9-x^2-y^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{9-x^2-y^2}}.$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(9-x^2-y^2)^{-1/2} \cdot (-2y) = \frac{-y}{\sqrt{9-x^2-y^2}}$$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(2,2)} = \frac{-2}{\sqrt{9-2^2-2^2}} = -2, \quad \left.\frac{\partial f}{\partial y}\right|_{(2,2)} = \frac{-2}{\sqrt{9-2^2-2^2}} = -2$$

$$\Rightarrow \vec{\nabla} f(2,2) = \langle -2, -2 \rangle$$ } so this is a vector pointing in the direction of steepest slope.

$$\therefore \|\vec{\nabla} f(2,2)\| = \|\langle -2,-2 \rangle\| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8}$$ } steepest slope

So the magnitude of the steepest slope is $\sqrt{8}$, and $\langle -2, -2 \rangle$ points in the direction of the steepest slope.