Lecture Examples - Monday July 24th

Telescopic Series:

1. Calculate $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$, if it converges.

Solution: Look at the N^{th} partial sum:

$$S_N = \underbrace{\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)}_{n=1} + \underbrace{\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)}_{n=2} + \underbrace{\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)}_{n=3} + \underbrace{\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)}_{n=4} +$$

$$\dots + \underbrace{\left(\cancel{\frac{1}{N-2}}-\cancel{\frac{1}{N}}\right)}_{n=N-2} + \underbrace{\left(\cancel{\frac{1}{N-1}}-\frac{1}{N+1}\right)}_{n=N-1} + \underbrace{\left(\cancel{\frac{1}{N}}-\frac{1}{N+2}\right)}_{n=N}$$

$$= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

$$\therefore \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1+\frac{1}{2}-\overbrace{\frac{1}{N+1}}^{0}-\overbrace{\frac{1}{N+2}}^{0}\right)$$

$$= \frac{3}{2}$$

Divergence Test:

1. Which of the following converge?

(a) $\sum_{n=1}^{\infty} \pi^{\pi}\left(1-\frac{1}{e^{e^{e}}}\right)^{n-1}$

Solution: $\sum_{n=1}^{\infty} \pi^{\pi}\left(1-\frac{1}{e^{e^e}}\right)^{n-1}$ is a geometric series with $r = 1-\frac{1}{e^{e^e}}$.

Since $|r|<1$, we know that it must be convergent. (To determine if it converges or diverges, we just need to find r, not a — we only need to determine a if we're asked to find the sum of the series.)

(b) $\sum_{n=1}^{\infty} \frac{n^2}{56n^2+9}$

Solution: Since $\lim_{n\to\infty} \frac{n^2}{56n^2+9} = \frac{1}{56} \neq 0$, by Divergence Test, this series must diverge.

(c) $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

Solution: Since this is NOT a geometric series and $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$, we cannot yet determine whether the series converges or diverges (the Divergence Test is inconclusive in this case).

Comparison Test

① Show that $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^{n-1}}$ converges.

Solution: Since $\frac{1}{n \cdot 2^{n-1}} \leq \frac{1}{2^{n-1}}$ for all n,

and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ is a geometric

series with $|r| = 1/2 < 1$ (and $\therefore$ converges),

by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^{n-1}}$ also converges.

② Determine whether the following converges or diverges

(a) $\sum_{n=10}^{\infty} \frac{\pi}{5n-4}$

Idea: When n is large, $\frac{\pi}{5n-4} \approx \frac{\pi}{5n}$,

so we'd expect $\sum_{n=10}^{\infty} \frac{\pi}{5n-4}$ to converge/diverge like $\sum_{n=10}^{\infty} \frac{\pi}{5n}$. Now $\sum_{n=10}^{\infty} \frac{\pi}{5n} = \frac{\pi}{5} \sum_{n=10}^{\infty} \frac{1}{n}$

is a p-series with $p=1$ and $\therefore$ diverges.
So, we'd expect $\sum_{n=10}^{\infty} \frac{\pi}{5n-4}$ to diverge and we should try to compare it to $\sum_{n=10}^{\infty} \frac{\pi}{5n}$.

This is NOT a solution, just intuition!

Solution: Since $\frac{\pi}{5n-4} \geq \frac{\pi}{5n}$ for all n and $\sum_{n=10}^{\infty} \frac{\pi}{5n} = \frac{\pi}{5} \sum_{n=10}^{\infty} \frac{1}{n}$ is divergent (p-series with $p=1$), it follows from the Comparison Test that $\sum_{n=10}^{\infty} \frac{\pi}{5n-4}$ also diverges.

** For a complete solution, you need to

(i) Verify the inequality.

(ii) Justify why $\sum_{n=1}^{\infty} b_n$ converges or diverges. (In this case $\sum_{n=10}^{\infty} b_n = \sum_{n=10}^{\infty} \frac{\pi}{5n}$).

(iii) State the Test being used. (in this case, Comparison Test.) **

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{10^{10}}{e^n+n}$

Idea: When n is large $\displaystyle\frac{10^{10}}{e^n+n} \approx \frac{10^{10}}{e^n}$ (the exponential e^n dominates the polynomial n) and $\displaystyle\sum_{n=2}^{\infty} \frac{10^{10}}{e^n} = \sum_{n=2}^{\infty} 10^{10} \cdot \left(\frac{1}{e}\right)^n$ is a geometric series with $r = \frac{1}{e} < 1$ (note: $a \neq 10^{10}$, but that's not important). Since $|r| < 1$, the geometric series converges, so we'd expect $\displaystyle\sum_{n=2}^{\infty} \frac{10^{10}}{e^n+n}$ to also converge.

Solution: We have that $\displaystyle\frac{10^{10}}{e^n+n} \leq \frac{10^{10}}{e^n}$ for all n. So, since $\displaystyle\sum_{n=2}^{\infty} \frac{10^{10}}{e^n}$ is a convergent series (because it's geometric with $|r| = 1/e < 1$), it follows from the Comparison Test that $\displaystyle\sum_{n=2}^{\infty} \frac{10^{10}}{e^n+n}$ also <u>converges</u>.

Limit Comparison Test

① Show that $\displaystyle\sum_{n=10}^{\infty} \frac{\pi}{5n+4}$ diverges.

Idea: When n is large, $\dfrac{\pi}{5n+4} \approx \dfrac{\pi}{5n}$ and (~~fmo~~ as earlier) we know that the series $\displaystyle\sum_{n=10}^{\infty} \frac{\pi}{5n}$ diverges. However

$$\underbrace{\frac{\pi}{5n+4} \leq \frac{\pi}{5n}}$$

The inequality goes the wrong way $\Rightarrow$ can't use regular Comparison Test.

* Try Limit Comparison with $a_n = \dfrac{\pi}{5n+4}$, $b_n = \dfrac{\pi}{5n}$.

Solution: We have that

$$\lim_{n\to\infty} \frac{\left(\frac{\pi}{5n+4}\right)}{\left(\frac{\pi}{5n}\right)} = \lim_{n\to\infty} \frac{5n}{5n+4} = \lim_{n\to\infty} \frac{5}{5+4/n} = 1 >$$

so, since $\displaystyle\sum_{n=10}^{\infty} \frac{\pi}{5n} = \frac{\pi}{5}\sum_{n=10}^{\infty} \frac{1}{n}$ diverges. (p-series with $p=1$), it follows from the Limit Comparison Test that $\displaystyle\sum_{n=10}^{\infty} \frac{\pi}{5n+4}$ also diverges.

② Determine if the series $\sum_{n=17}^{\infty} \frac{2}{n^{3/2}-1}$ converges or diverges.

Idea: When n is large, $\frac{2}{n^{3/2}-1} \approx \frac{2}{n^{3/2}}$

and $\sum_{n=17}^{\infty} \frac{2}{n^{3/2}} = 2 \cdot \sum_{n=17}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = 3/2 > 1$ (and $\therefore$ convergent). However

$$\underbrace{\frac{2}{n^{3/2}-1} \geq \frac{2}{n^{3/2}}}_{\text{Inequality goes the wrong way}}$$

→ Try Limit Comparison Test.

Solution: We have that

$$\lim_{n\to\infty} \frac{\left(\frac{2}{n^{3/2}-1}\right)}{\left(\frac{2}{n^{3/2}}\right)} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}-1} = 1 > 0.$$

So, since $\sum_{n=17}^{\infty} \frac{2}{n^{3/2}} = 2 \sum_{n=17}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series ($p = 3/2 > 1$), it follows from the Limit Comparison Test that $\sum_{n=17}^{\infty} \frac{2}{n^{3/2}-1}$ also converges.